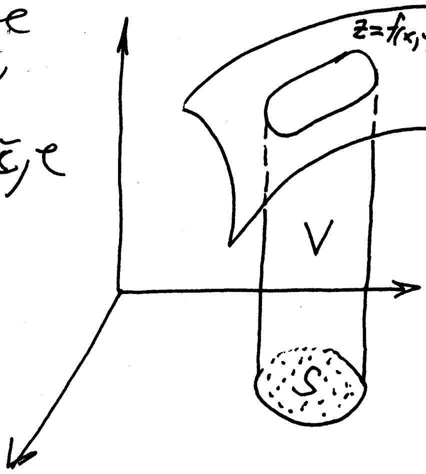
Primjena dvostrukog integrala

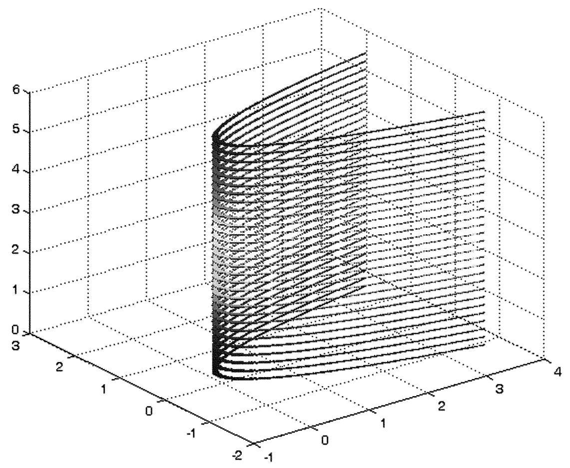
1° Površina zatvorene i ograničene oblasti D

$$P=\iint_D dx\,dy$$

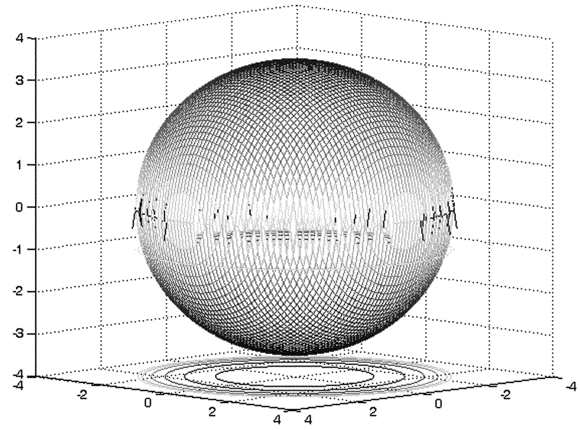
2° Zapremina tijela koje odozgo određuje površ $z=f(x,y)$, odozdo ravan $z=0$ a postranice valjkasta ploha koja na ravni XOY izrezuje omeđeno zatvoreno područje S iznosi



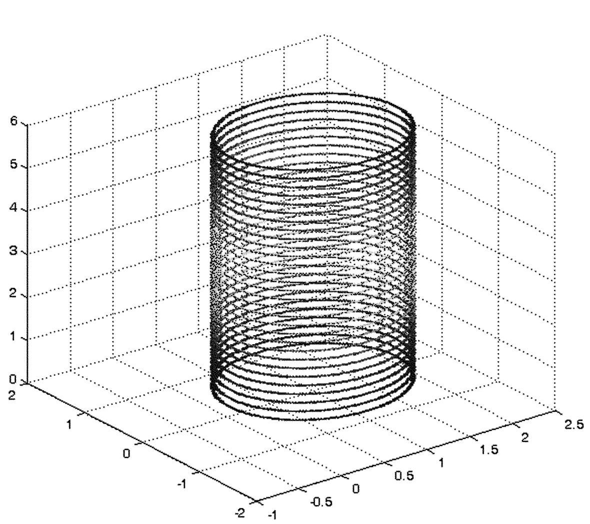
$$V=\iint_S f(x,y)\,dx\,dy$$



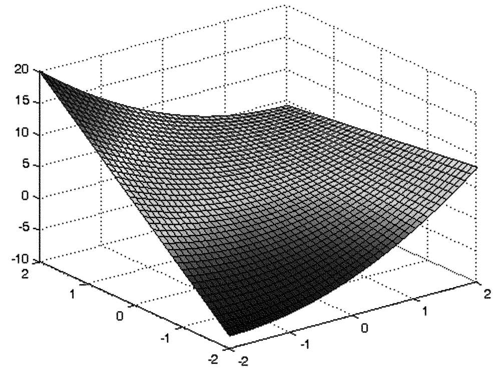
cilindar $x=2y^2$



kugla $x^2+y^2+z^2=12$



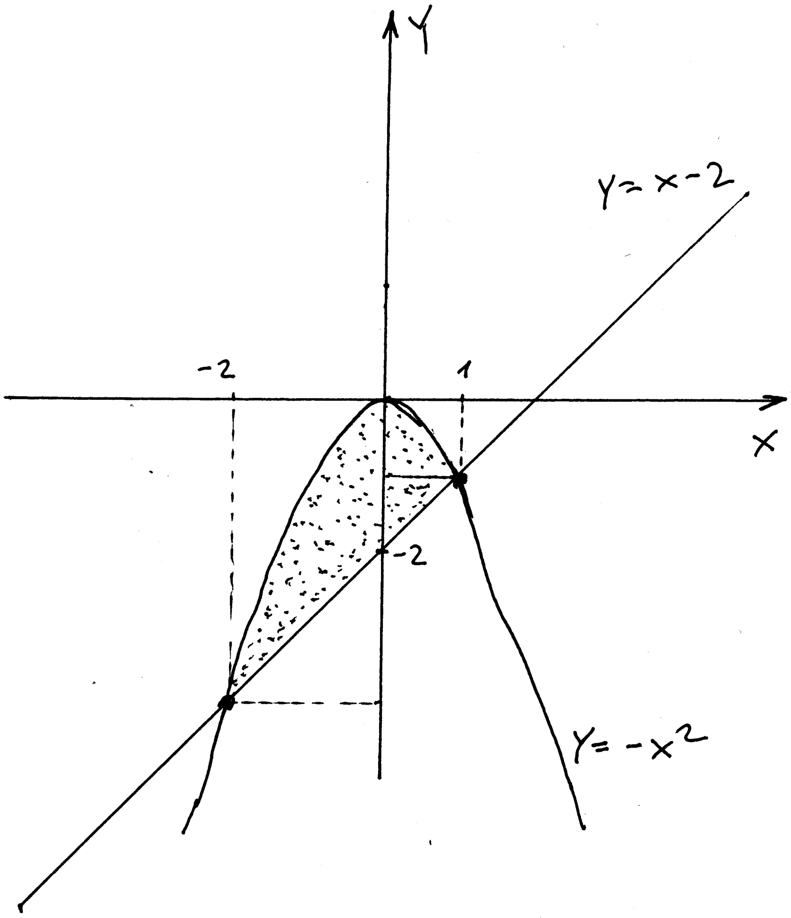
valjak $x^2+y^2=2x$



funkcija $z=x^2-2xy+3y+2$

(#) Naći površinu figure ograničene linijama $y=-x^2$, $x-y-2=0$.

Rj. Nacrtajmo sliku



Pronađimo presječne tačke krive $y=-x^2$ i prave $x-y-2=0$.

$$\frac{\begin{array}{l} y=-x^2 \\ x-y-2=0 \end{array}}{x+x^2-2=0}$$

$$x^2+x-2=0$$

$$D=1+8=9 \qquad x_{1,2}=\frac{-1\pm 3}{2}$$

$$x_1=-2, \quad x_2=1$$

$$(x-1)(x+2)=0$$

$$x=1 \Rightarrow y=-1$$

$$x=-2 \Rightarrow y=-4$$

I način:

$$P=\int_{-2}^{1}\left(-x^2-(x-2)\right)dx=\int_{-2}^{1}(-x^2-x+2)\,dx=\overbrace{-\frac{1}{3}x^3\Big|_{-2}^{1}}^{1+8}\overbrace{-\frac{1}{2}x^2\Big|_{-2}^{1}}^{1-4}\overbrace{+2x\Big|_{-2}^{1}}^{1+2}=$$

$$=-\frac{1}{3}\cdot 9+\frac{1}{2}\cdot 3+2\cdot 3=-3+\frac{3}{2}+6=-3-\frac{3}{2}=\frac{9}{2}$$

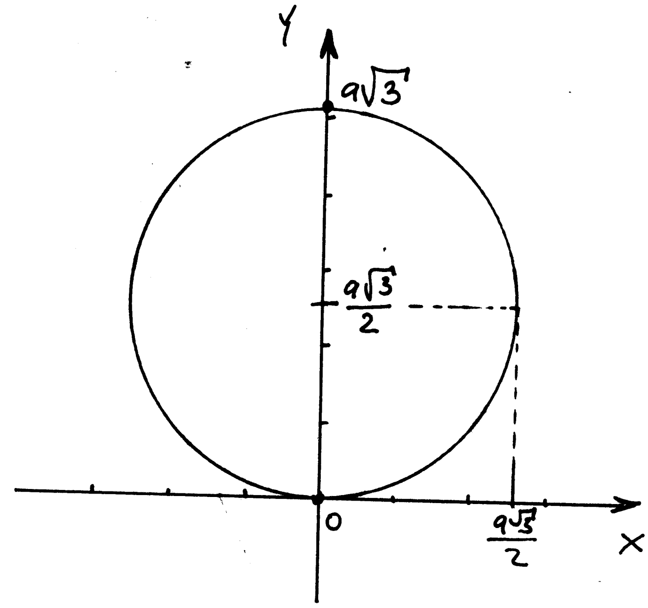
II način:

$$P=\iint_D dx\,dy \quad \text{gdje je} \quad D:\begin{cases}-2\le x\le 1\\ x-2\le y\le -x^2\end{cases}$$

$$P=\iint_D dx\,dy=\int_{-2}^{1}dx\int_{x-2}^{-x^2}dy=\int_{-2}^{1}\left((-x^2)-(x-2)\right)dx=\dots=\frac{9}{2}$$

(#) Izračunati površinu figure koja je ograničena linijom $x^2+y^2=a\sqrt{3}\,y$.

Rj: $P=\iint\limits_D dx\,dy$



$x^2+y^2=a\sqrt{3}\,y$

$x^2+y^2-a\sqrt{3}\,y=0$

$x^2+y^2-2\cdot\frac{a\sqrt{3}}{2}y+\frac{a^2\cdot 3}{4}-\frac{3a^2}{4}=0$

$x^2+\left(y-\frac{a\sqrt{3}}{2}\right)^2=\left(\frac{a\sqrt{3}}{2}\right)^2$

krug s centrom u tački $C\left(0,\frac{a\sqrt{3}}{2}\right)$ poluprečnika $\frac{a\sqrt{3}}{2}$.

Uvodim smjene $x=r\cos\varphi$ $\quad 0\le r\le\frac{a\sqrt{3}}{2}$

$y=\frac{a\sqrt{3}}{2}+r\sin\varphi$ $\quad 0\le\varphi\le 2\pi$

$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}$ $\quad \frac{\partial x}{\partial r}=\cos\varphi \quad \frac{\partial x}{\partial\varphi}=-r\sin\varphi$ $\quad dx\,dy=|J|\,dr\,d\varphi$

$\frac{\partial y}{\partial r}=\sin\varphi \quad \frac{\partial y}{\partial\varphi}=r\cos\varphi$ $\quad J=r$

$P=\iint\limits_D dx\,dy=\iint\limits_{D'}|r|\,dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{a\sqrt{3}}{2}} r\,dr\right]d\varphi=$

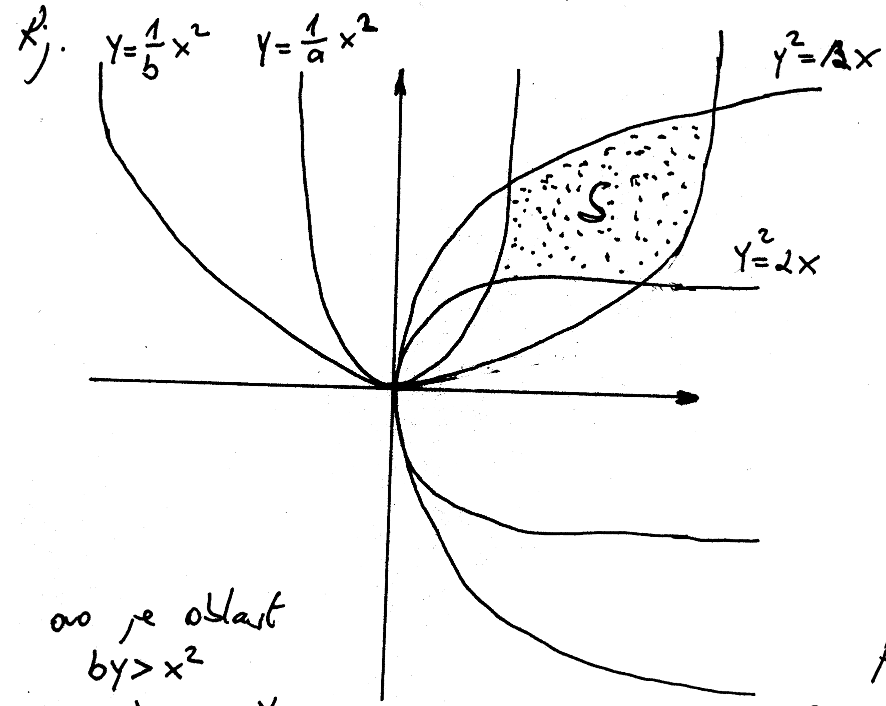
$=\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^{\frac{a\sqrt{3}}{2}}d\varphi=\frac{a^2\cdot 3}{4}\cdot\frac{1}{2}\varphi\Big|_0^{2\pi}=\frac{3a^2}{4}\cdot\pi$

površina figure koja je ograničena linijom

\# Izračunati površinu krivolinijskog 4-ugla omeđenog lukovima parabola $x^2 = ay$, $x^2 = by$, $y^2 = \alpha x$ i $y^2 = \beta x$ $(0 < a < b,\ 0 < \alpha < \beta)$.

$$P = \iint dx\,dy$$

R.



$x^2 = ay$, $\quad S \quad x^2 = by$

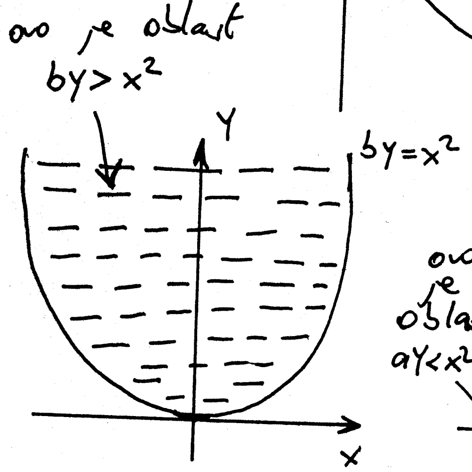
$y = \frac{1}{a}x^2 \qquad y = \frac{1}{b}x^2$

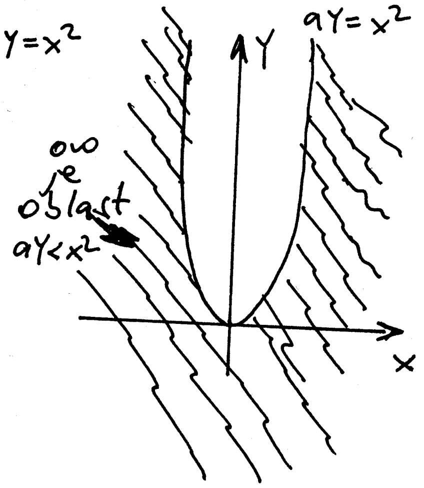
$a < b \qquad \frac{1}{a}x^2 > \frac{1}{b}x^2$

$\frac{1}{a} > \frac{1}{b}$

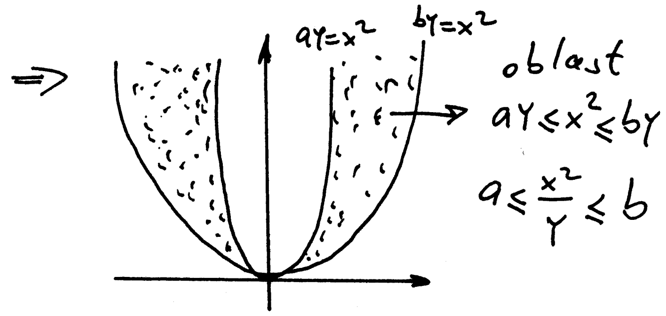
Na klasičan način površinu $\iint\limits_S dx\,dy$ je teško izračunati.

Primjetimo sljedeće:





$\Rightarrow$



Slično $y^2 \geq \alpha x$ i $y^2 \leq \beta x$

imamo $\alpha x \leq y^2 \leq \beta x \qquad \alpha \leq \frac{y^2}{x} \leq \beta$

Vidimo da možemo uvesti smjene $u = \frac{x^2}{y}$ i $v = \frac{y^2}{x}$ gdje

$a \le u \le b \qquad u = \frac{x^2}{y} \qquad v = \frac{y^2}{x}$

$\alpha \le v \le \beta \qquad y = \frac{x^2}{u}. \qquad x = \frac{y^2}{v} \quad \Rightarrow \quad x = \frac{\left(\frac{x^2}{u}\right)^2}{v} = \frac{x^4}{u^2 v} \quad \Rightarrow \quad x^3 = u^2 v$

$x = \sqrt[3]{u^2 v}$

$$y = \frac{x^2}{u} = \frac{\sqrt[3]{(u^2v)^2}}{u} = \sqrt[3]{\frac{u^4v^2}{u^3}} = \sqrt[3]{uv^2} \qquad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad dx\,dy = |J|\,du\,dv$$

$x = u^{\frac{2}{3}} v^{\frac{1}{3}}, \qquad \frac{\partial x}{\partial u} = \frac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}} \qquad \frac{\partial x}{\partial v} = u^{\frac{2}{3}}\frac{1}{3}v^{-\frac{2}{3}}$

$y = u^{\frac{1}{3}} v^{\frac{2}{3}} \qquad \frac{\partial y}{\partial u} = \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}} \qquad \frac{\partial y}{\partial v} = u^{\frac{1}{3}}\frac{2}{3}v^{-\frac{1}{3}} \qquad J = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$

$$\iint\limits_S dx\,dy = \int_a^b \left[\int_\alpha^\beta \frac{1}{3}\,dv\right] du = \frac{1}{3}\int_a^b v\Big|_\alpha^\beta du = \frac{1}{3}(\beta-\alpha)u\Big|_a^b = \frac{1}{3}(b-a)(\beta-\alpha)$$

Izračunati zapreminu tijela, ograničeno površinama

$y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

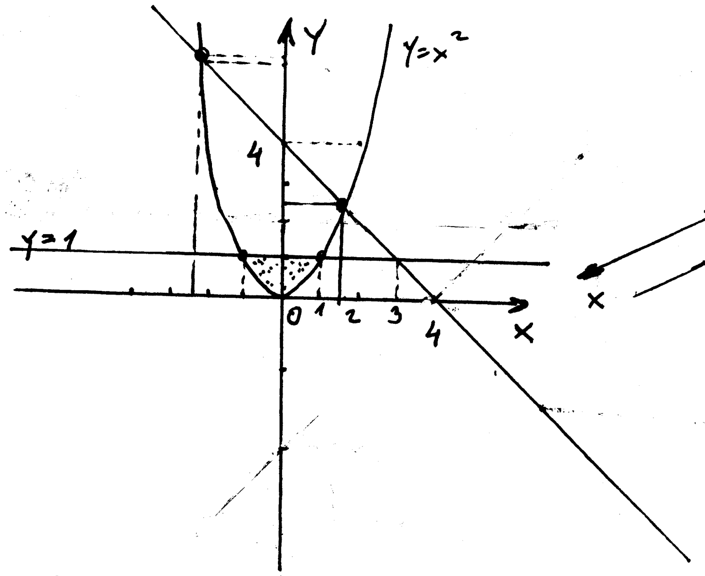
Rj. Skicirajmo naše tijelo.

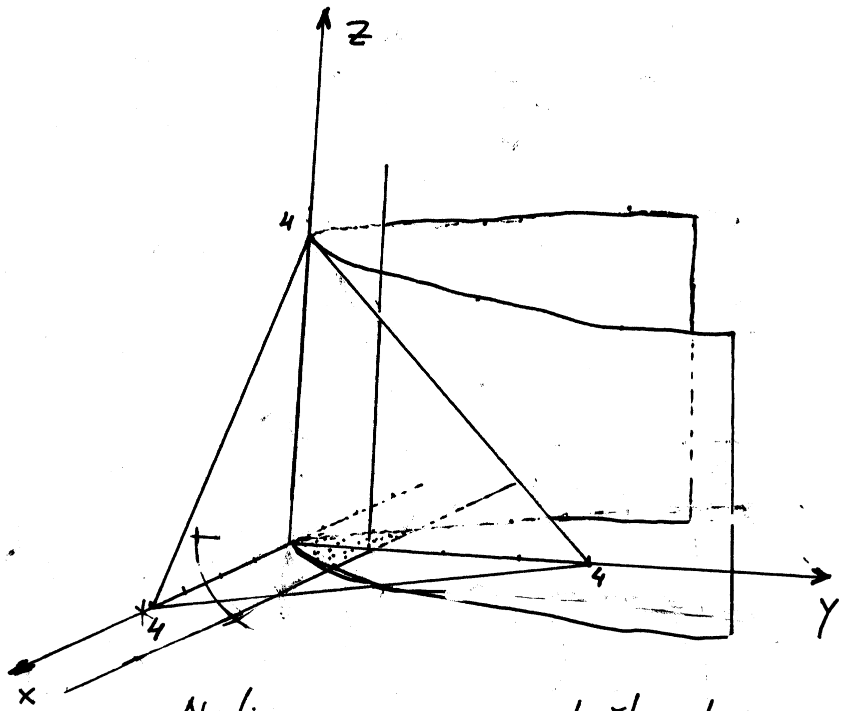
$x+y+z=4$ je ravan $\left(\frac{x}{4}+\frac{y}{4}+\frac{z}{4}=1\right)$ koja na x, y, i z osi ima odsječke 4.

$y=1$, $z=0$ su ravni

$y=x^2$ je cilindar 

Napravimo ortogonalne projekcije površina na xOy ravan





Nađimo presječnu tačku krive $y=x^2$ i prave $x+y=4$.

$$\frac{\begin{array}{l} y=x^2 \\ x+y=4 \end{array}}{\begin{array}{l} y=x^2 \\ y=4-x \end{array}}$$

$x^2=4-x$

$x^2+x-4=0$

$D=1+16=17$

$x_{1,2}=\frac{-1\pm 4{,}12}{2}$

$x_1 \approx -2{,}56$ $\Downarrow$ $y_1=2{,}43$

$x_2 \approx 1{,}56$ $\Downarrow$ $y_2=6{,}56$

$V=\iint_D f(x,y)\,dx\,dy$ ← zapremina tijela koje je odozgo ograničeno sa i tijelo ima ortogonalnu projekciju D

ovo

U našem slučaju $f(x,y)=4-x-y$ (vidimo sa skice)

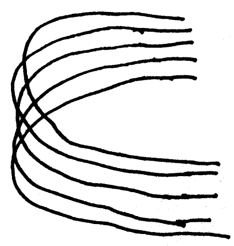
$V=\iint_D (4-x-y)\,dx\,dy$ gdje je $D:\begin{cases} -1\le x\le 1 \\ x^2\le y\le 1 \end{cases}$ ili $D:\begin{cases} 0\le y\le 1 \\ -\sqrt{y}\le x\le \sqrt{y} \end{cases}$

$$V=\int_{-1}^{1}dx\int_{x^2}^{1}(4-x-y)\,dy=\int_{-1}^{1}\Big(\underbrace{4y\Big|_{x^2}^{1}}_{4(1-x^2)}-\underbrace{xy\Big|_{x^2}^{1}}_{-x(1-x^2)}\underbrace{-\frac{1}{2}y^2\Big|_{x^2}^{1}}_{-\frac{1}{2}(1-x^4)}\Big)\,dx=$$

$$=\int_{-1}^{1}\left(4-4x^2-x+x^3-\frac{1}{2}+\frac{1}{2}x^4\right)dx=\int_{-1}^{1}\left(x^3-4x^2+\frac{1}{2}x^4-x+\frac{7}{2}\right)=\dots=-\frac{8}{3}+\frac{1}{5}+7=\underbrace{\frac{68}{15}}_{\text{traženo rješenje}}$$

Izračunati zapreminu tijela koje je ograničeno površinama $x=2y^2$, $x+2y+z=4$ i $z=0$.

Rj. $x=2y^2$ cilindar u prostoru



Pronađimo projekciju površina na xO_y ravan:

$$\begin{array}{l} x=2y^2 \\ x+2y=4 \\ \hline 2y^2+2y=4 \quad /:2 \end{array}$$

$$y^2+y-2=0$$

$$(y-1)(y+2)=0$$

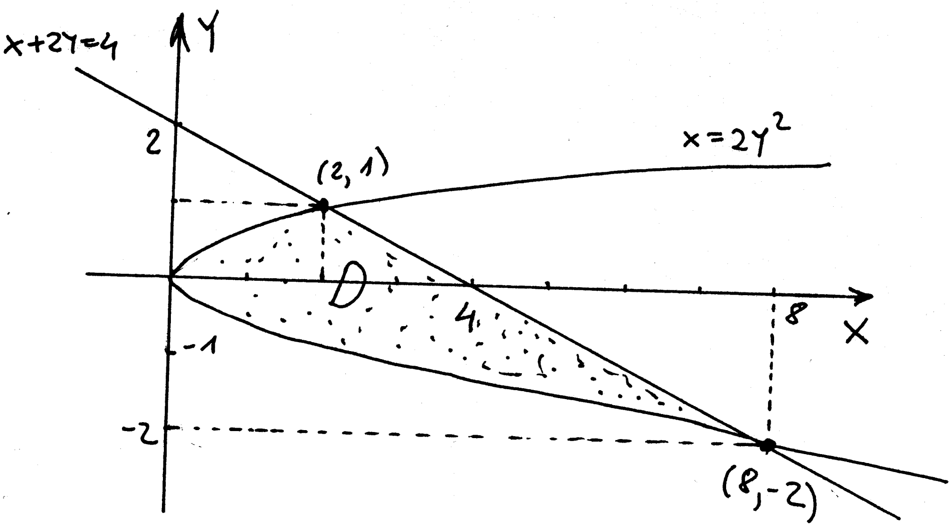
$$y_1=1 \Rightarrow x_1=2$$

$$y_2=-2 \Rightarrow x_2=8$$

$$x+2y=4 \quad /:4$$

$$\frac{x}{4}+\frac{y}{2}=1$$

Nacrtajmo sliku



$$D: \begin{cases} -2 \le y \le 1 \\ 2y^2 \le x \le 4-2y \end{cases}$$

$$x+2y+z=4$$

$$z=4-x-2y$$

$$V=\iint_D (4-x-2y)\,dx\,dy$$

$$V=\int_{-2}^{1}\left[\int_{2y^2}^{4-2y}(4-x-2y)\,dx\right]dy=\int_{-2}^{1}\left[4x\Big|_{2y^2}^{4-2y}-\frac{1}{2}x^2\Big|_{2y^2}^{4-2y}-2y\cdot x\Big|_{2y^2}^{4-2y}\right]dy=$$

$$=\int_{-2}^{1}\Big[\underbrace{4(4-2y-2y^2)}_{16-8y-8y^2}-\underbrace{\frac{1}{2}\left((4-2y)^2-(2y^2)^2\right)}_{16-16y+4y^2-4y^4}-2y(4-2y-2y^2)\Big]dy$$

$$=\int_{-2}^{1}\left[16-\cancel{8y}-8y^2-8+\cancel{8y}-2y^2+2y^4-8y+4y^2+4y^3\right]dy=\int_{-2}^{1}(2y^4-6y^2+4y^3-8y+8)\,dy$$

$$=\frac{2}{5}y^5\Big|_{-2}^{1}-\frac{6}{3}y^3\Big|_{-2}^{1}+\frac{4}{4}y^4\Big|_{-2}^{1}-\frac{8}{2}y^2\Big|_{-2}^{1}+8y\Big|_{-2}^{1}=\frac{2}{5}\cdot 33-2\cdot 9+1\cdot(-15)-\frac{8}{2}(-3)$$

$$+8\cdot 3=\frac{66}{5}-18-15+12+24=\frac{66}{5}+36-33=\frac{66}{5}+\frac{15}{5}=\frac{81}{5}$$

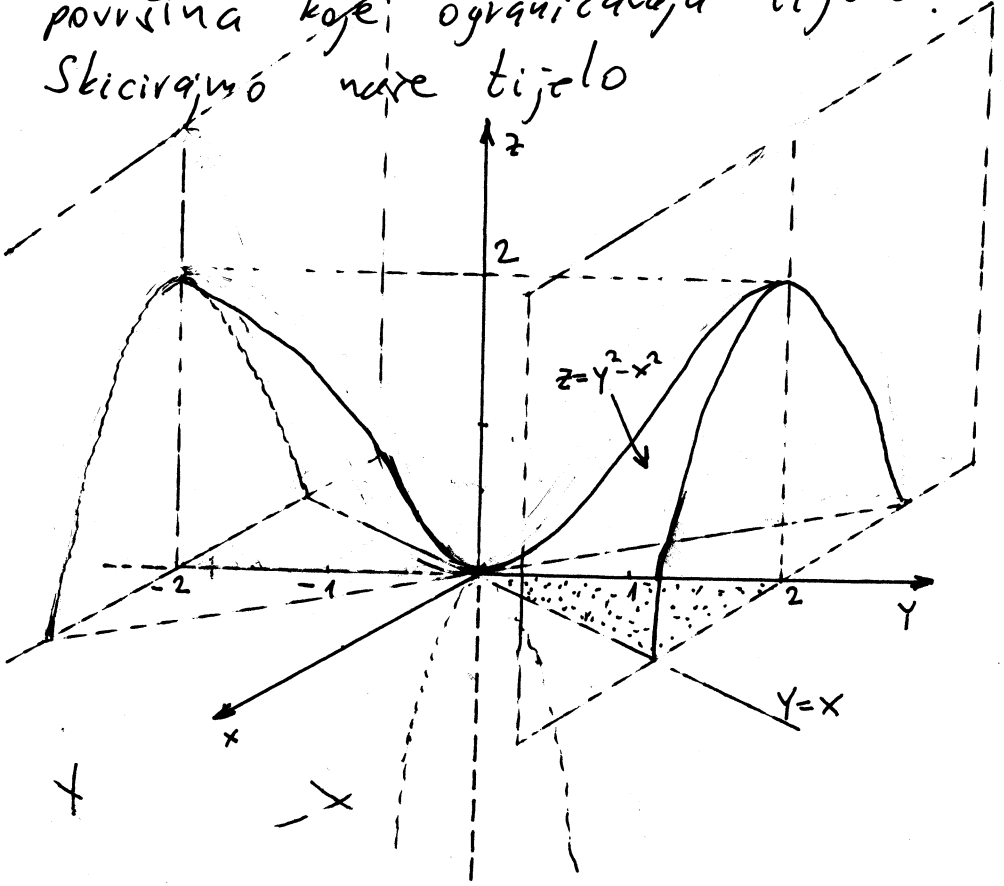
(#) Izračunati zapreminu tijela, koje je ograničeno sa površinama $z=y^2-x^2$, $z=0$, $y=\pm 2$.

Rj. Zapremina tijela se može računati pomoću dvostrukog ili pomoću trostrukog integrala. Za ta dva slučaja koristimo sljedeće dvije formule

$$V=\iint_D f(x,y)\,dx\,dy, \qquad V=\iiint_\Omega dx\,dy\,dz.$$

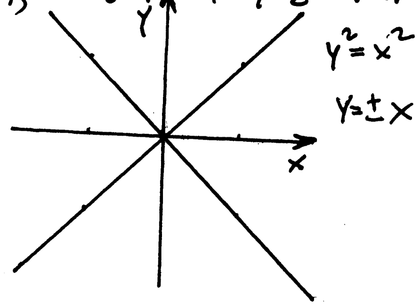
Koju od ove dvije formule je pogodnije koristiti zavisi od jednačina površina koje ograničavaju tijelo.

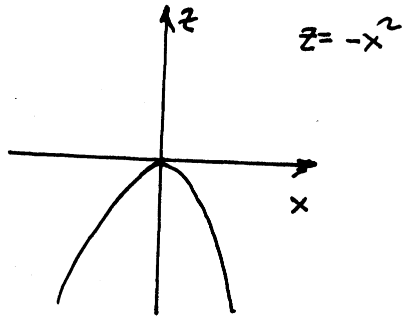
Skicirajmo naše tijelo

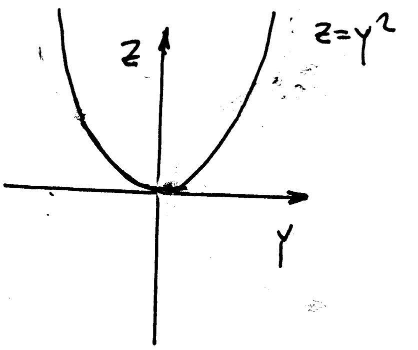


Šta predstavlja jednačina $z=y^2-x^2$?

Napravimo presjeke $z=y^2-x^2$ sa xOy, xOz i sa yOz ravni







$z(-x,-y)=(-y)^2-(-x)^2=y^2-x^2=z(x,y)$

$\Rightarrow$ tijelo je simetrično u odnosu na koordinatni početak

$z(x,-y)=(-y)^2-x^2=y^2-x^2$

$\Rightarrow$ tijelo je simetrično u odnosu na xOz osu

$z(-x,y)=y^2-(-x)^2=y^2-x^2 \Rightarrow$ tijelo je simetrično u odnosu na yOz -osu

Sa slike vidimo da možemo izabrati formulu za računanje zapremine $V = \iint\limits_D f(x,y)\,dx\,dy$ i to

$$V = 4\iint\limits_D (y^2 - x^2)\,dx\,dy \quad \text{gdje je} \quad D: \begin{cases} 0 \le y \le 2 \\ 0 \le x \le y \end{cases} \quad \text{(vidi sliku)}$$

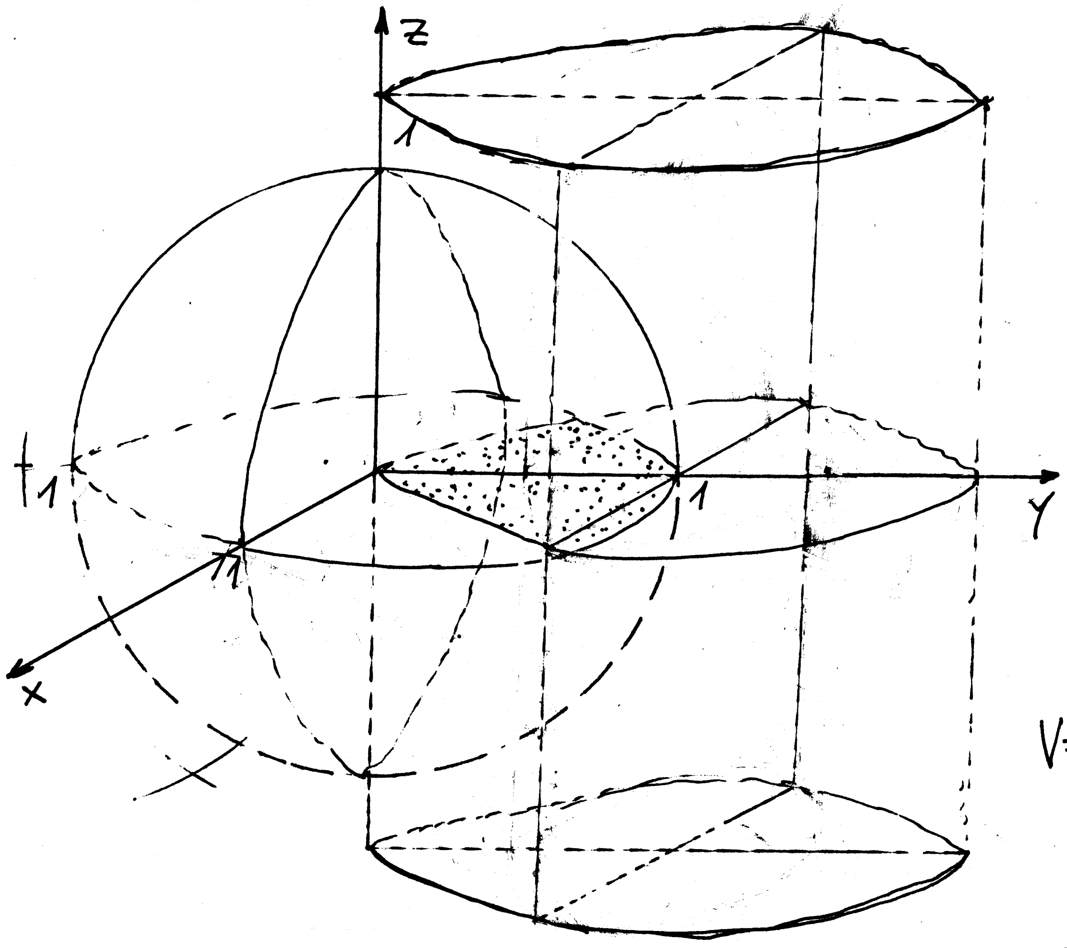
$$V = 4\int_0^2 dy \int_0^y (y^2 - x^2)\,dx = 4\int_0^2 \left(y^2 x \Big|_0^y - \frac{1}{3}x^3 \Big|_0^y \right) dy =$$

$$= 4\int_0^2 \left(y^3 - \frac{1}{3}y^3\right) dy = 4\int_0^2 \frac{2}{3}y^3\,dy = \frac{8}{3}\cdot\frac{1}{4}y^4 \Big|_0^2 = \frac{8}{3}\cdot\frac{1}{\cancel{4}_1}\cancel{16}^4 = \frac{32}{3}$$

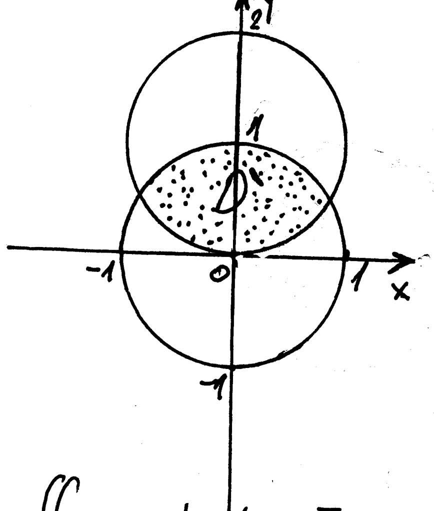
traženo rješenje

Izračunati zapreminu onog dijela lopte $x^2+y^2+z^2=1$ koji se nalazi unutar cilindra $x^2+(y-1)^2=1$.

Rj. Nacrtajmo skicu ove dvije figure u prostoru

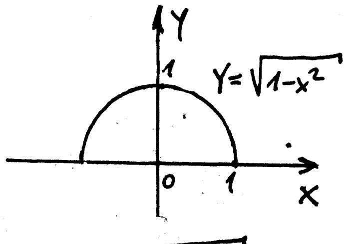


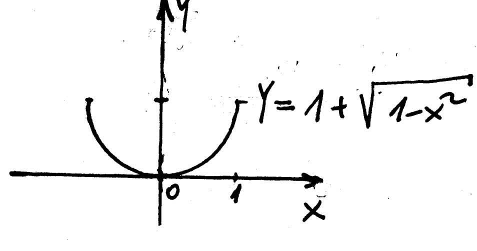
projekcije na xOy osu

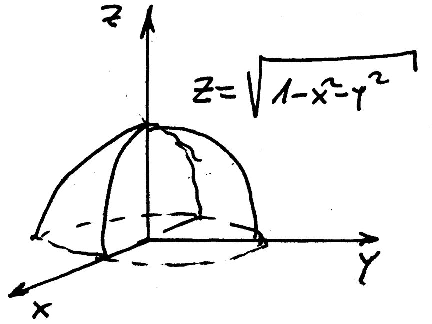


$$V=\iint_D z(x,y)\,dx\,dy$$ zapremina tijela koji je odozgo ograničen sa površi z a čija je projekcija na xOy ravan oblast D

Primjetimo da je presjek cilindra i lopte prvo simetričan u odnosu na xOy osu, a drugo da je simetričan u odnosu na yOz osu.







$$\frac{1}{4}V=\int_0^1 dx\int_{1+\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy$$

$$\frac{1}{2}V=\iint_{D'} z(x,y)\,dx\,dy,\quad D':\begin{cases}-1\le x\le 1\\ 1+\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\end{cases}$$

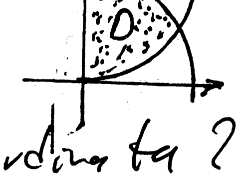
uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

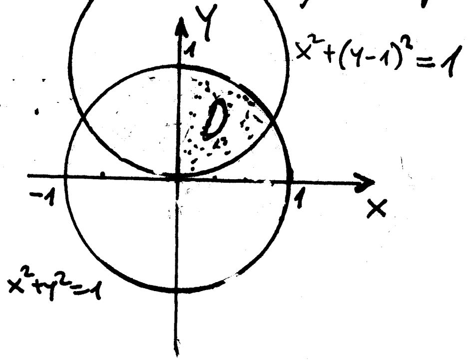
$dx\,dy=r\,dr\,d\varphi$

Kako opisati oblast



uz pomoć polarnih koordinata?

Opišimo unutrašnjost presjeka dva kruga pomoću polarnih koordinata.



$$x^2+y^2\le 1$$
$$(r\cos\varphi)^2+(r\sin\varphi)^2\le 1$$
$$r^2\le 1$$
$$0\le r\le 1$$

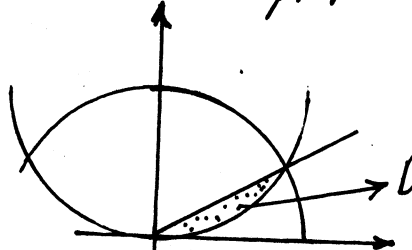
$$x^2+(y-1)^2\le 1$$
$$x^2+y^2-2y+1\le 1$$
$$x^2+y^2\le 2y$$
$$r^2\le 2r\sin\varphi \quad /:r$$
$$r\le 2\sin\varphi$$
$$0\le r\le 2\sin\varphi$$

Kako je $0\le\sin\varphi\le 1$ (ako posmatramo prvi kvadrant) to je moguć i slučaj da je $2\sin\varphi>1$ pa imamo dva slučaja

1° $2\sin\varphi\le 1 \Rightarrow \sin\varphi\le\frac{1}{2}$ (pa ako posmatramo $(\sin\varphi\ge 0)$ prvi kvadrant)

$\Rightarrow \varphi\in\left(0,\frac{\pi}{6}\right)$

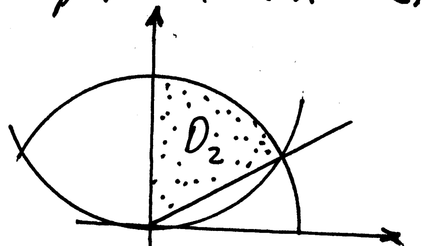
$$D_1:\begin{cases}0\le\varphi\le\frac{\pi}{6}\\ 0\le r\le 2\sin\varphi\end{cases}$$



2° $2\sin\varphi\ge 1 \Rightarrow \sin\varphi\ge\frac{1}{2}$ (pa za prvi kvadrant $\sin\varphi\le 1$)

$\Rightarrow \varphi\in\left(\frac{\pi}{6},\frac{\pi}{2}\right)$

$$D_2:\begin{cases}\frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 1\end{cases}$$



$D=D_1\cup D_2$

$$\frac{1}{4}V=\iint\limits_D\sqrt{1-r^2}\,r\,dr\,d\varphi=\iint\limits_{D_1}r\sqrt{1-r^2}\,dr\,d\varphi+\iint\limits_{D_2}r\sqrt{1-r^2}\,dr\,d\varphi$$

$$\iint\limits_{D_1}r\sqrt{1-r^2}\,dr\,d\varphi=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}r\sqrt{1-r^2}\,dr=\left|\begin{array}{l}d(1-r^2)=\\=-2r\,dr\end{array}\right|=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}\left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\sin\varphi}d\varphi=-\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\int_0^{\frac{\pi}{6}}\left((1-\underbrace{4\sin^2\varphi}_{(2\sin\varphi)^2})^{\frac{3}{2}}-1\right)d\varphi$$

Ovo je eliptički integral i on se ne mora izračunati. Njegova približna vrijednost je $\pi/18$.

$$\iint\limits_{D_2}r\sqrt{1-r^2}\,dr\,d\varphi=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}d\varphi\int_0^1\left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1d\varphi=-\frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(-1)\,d\varphi$$

$$=\frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{1}{3}\cdot\frac{3\pi-\pi}{6}=\frac{1}{3}\cdot\frac{\overset{1}{\cancel{2}}\pi}{\underset{3}{\cancel{6}}}=\frac{\pi}{9}$$

tražena zapremina

$$\frac{1}{4}V=\frac{\pi}{9}+\frac{\pi}{18}=\frac{3\pi}{18}=\frac{\pi}{6}\qquad V=\frac{4\pi}{6}=\frac{2\pi}{3}$$

Izračunati zapreminu tijela ograničenog površima:

104. $z=x^2+y^2,\ y=x^2,\ x=1,\ z=0.$

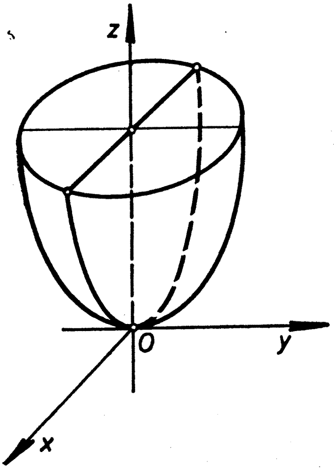
105. $z=xy,\ y=0,\ x=0,\ z=0,\ x^2+y^2=r^2.$

Rješenja:

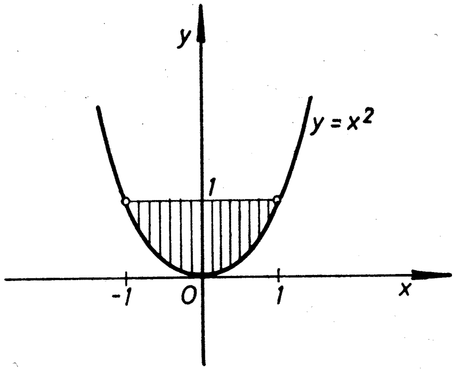
104. Zapremina tijela V ograničenog sa ravni $z=0$, površi $z=f(x, y)$ $(z\geqslant 0)$ i cilindrom koji izrezuje oblast D (x, y)-ravni, a ima izvodnice paralelne sa z-osom, data je sa

$$V=\iint_D f(x, y)\,dxdy.$$

U ovom slučaju površ $z=f(x, y)$ je paraboloid $z=x^2+y^2$, (slika 18a) dok je oblast D data na slici 18b.



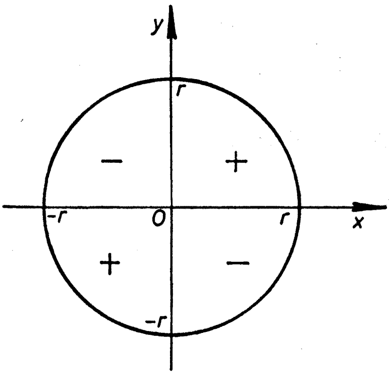
Sl. 18 a



Sl. 18 b

Biće

$$V=\iint_D (x^2+y^2)\,dxdy=\int_{-1}^{1} dx\int_{x^2}^{1}(x^2+y^2)\,dy=\frac{88}{105}.$$



Sl. 19

105. Tijelo V se sastoji iz četiri jednaka dijela od kojih su dva ispod ravni $z=0$ (sl. 19). Biće

$$V=4\int_0^r x\,dx\int_0^{\sqrt{r^2-x^2}} y\,dy=$$

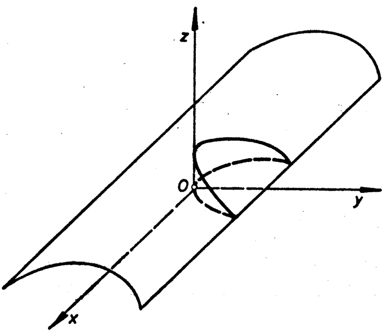
$$=2\int_0^r x\,(r^2-x^2)\,dx=\frac{r^4}{2}.$$

Izračunati zapreminu tijela ograničenog površima:

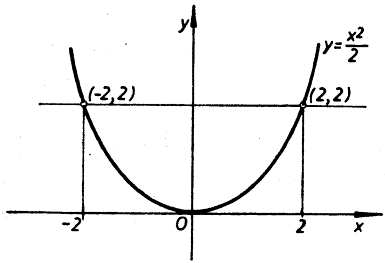
106. $z = 4 - y^2, \quad y = \dfrac{x^3}{2}, \quad z = 0.$

Rješenja:

106. Površ $z = 4 - y^2$ je parabolični cilindar okomit na ravan yOz, a površ $y = \dfrac{x^2}{2}$ je parabolični cilindar okomit na ravan xOy (sl. 20). Tijelo V projektuje se na oblast D u ravni $z = 0$ ograničenu parabolom $y = \dfrac{x^2}{2}$ i presjekom cilindra $z = 4 - y^2$ i ravni $z = 0$ (sl. 21).



Sl. 20



Sl. 21

$$V = \iint_D (4 - y^2)\,dx\,dy = \int_{-2}^{2} dx \int_{\frac{x^2}{2}}^{2} (4 - y^2)\,dy = 2\int_{0}^{2} dx \int_{\frac{x^2}{2}}^{2} (4 - y^2)\,dy =$$

$$= 2\int_{0}^{2} \left(4y - \frac{y^3}{3}\right)\Bigg|_{\frac{x^2}{2}}^{2} dx = 2\int_{0}^{2} \left(8 - \frac{8}{3} - 2x^2 + \frac{x^6}{24}\right) dx = \frac{256}{21}.$$

#

Izračunati zapreminu tijela ograničenog površima:

107. $z = 1 - 4x^2 - y^2, \; z = 0.$

Rješenja:

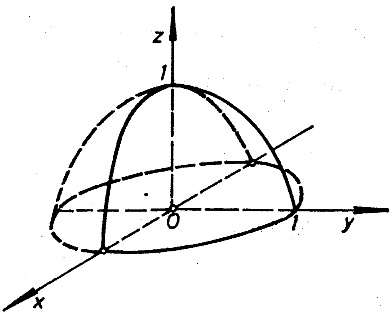
107. Paraboloid $z = 1 - 4x^2 - y^2$ je okrenut nadolje, i siječe se sa ravni $z = 0$ po elipsi $4x^2 + y^2 = 1$ (sl. 22 i sl. 23). Zato je

$$V = \iint_D (1 - 4x^2 - y^2) = \int_{-1/2}^{1/2} dx \int_{-\sqrt{1-4x^2}}^{\sqrt{1-4x^2}} (1 - 4x^2 - y^2)\, dy =$$

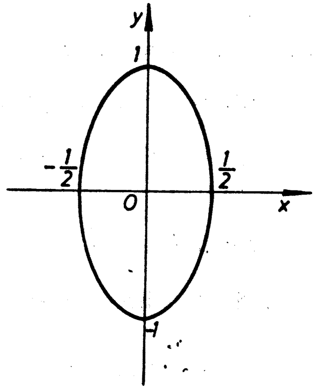
$$= 4\int_0^{1/2} dx \int_0^{\sqrt{1-4x^2}} (1 - 4x^2 - y^2) dy = \frac{8}{3}\int_0^{1/2} (1 - 4x^2)^{\frac{3}{2}}\, dx.$$

Smjenom $2x = \sin t$ dobija se

$$V = \frac{4}{3}\int_0^{\pi/2} \cos^4 t\, dt = \frac{4}{3}\int_0^{\pi/2} \left(\frac{1 + \cos 2t}{2}\right)^2 dt = \frac{\pi}{4}.$$



Sl. 22



Sl. 23

Zadaci za vježbu

Zapremina tela. I

U zadacima 3559 — 3596 pomoću dvojnih integrala naći zapremine tela ograničenih datim površima (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3559. Koordinatnim ravnima, ravnima $x=4$ i $y=4$ i obrtnim paraboloidom $z=x^2+y^2+1$.

3560. Koordinatnim ravnima, ravnima $x=a$, $y=b$ i eliptičnim paraboloidom $z=\frac{x^2}{2p}+\frac{y^2}{2q}$.

3561. Ravnima $x=0$, $y=0$, $z=0$ i $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (piramida).

3562. Ravnima $y=0$, $z=0$, $3x+y=6$, $3x+2y=12$ i $x+y+z=6$.

3563. Obrtnim paraboloidom $z=x^2+y^2$, koordinatnim ravnima i ravni $x+y=1$.

3564. Obrtnim paraboloidom $z=x^2+y^2$ i ravnima $z=0$, $y=1$, $y=2x$ i $y=6-x$.

3565. Cilindrima $y=\sqrt{x}$, $y=2\sqrt{x}$ i ravnima $z=0$ i $x+z=6$.

3566. Cilindrom $z=\frac{1}{2}y^2$ i ravnima $x=0$, $y=0$, $z=0$ i $2x+3y-12=0$.

3567. Cilindrom $z=9-y^2$, koordinatnim ravnima i ravni $3x+4y=12$ $(y\geqslant 0)$.

3568. Cilindrom $z=4-x^2$, koordinatnim ravnima i ravni $2x+y=4$ $(x\geqslant 0)$.

3569. Cilindrom $2y^2=x$ i ravnima $\frac{x}{4}+\frac{y}{2}+\frac{z}{4}=1$ i $z=0$.

3570. Kružnim cilindrom poluprečnika r, čija se osa poklapa sa ordinatnom osom, koordinatnim ravnima i ravni $\frac{x}{r}+\frac{y}{a}=1$.

3571. Eliptičnim cilindorom $\frac{x^2}{4}+y^2=1$ i ravnima $z=12-3x-4y$ i $z=1$.

3572. Cilindrima $x^2+y^2=R^2$ i $x^2+z^2=R^2$.

3573. Cilindrima $z=4-y^2$, $y=\frac{x^2}{2}$ i ravni $z=0$.

3574. Cilindrima $x^2+y^2=R^2$, $z=\frac{x^3}{a^2}$ i ravni $z=0$ $(x\geqslant 0)$.

3575. Hiperboličnim paraboloidom $z=x^2-y^2$ i ravnima $z=0$ i $x=3$.

3576. Hiperboličnim paraboloidom $z=xy$, cilindrom $y=\sqrt{x}$ i ravnima $x+y=2$, $y=0$ i $z=0$.

3577. Paraboloidom $z=x^2+y^2$, cilindrom $y=x^2$ i ravnima $y=1$ i $z=0$.

3578. Eliptičnim cilindrom $\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$ i ravnima $y=\frac{b}{a}x$, $y=0$ i $z=0$ $(x>0)$.

3579. Paraboloidom $z=\frac{a^2-x^2-4y^2}{a}$ i ravni $z=0$.

3580. Cilindrima $y=e^x$, $y=e^{-x}$, $z=e^2-y^2$ i ravni $z=0$.

3581. Cilindrima $y=\ln x$ i $z=\ln^2 x$ i ravnima $z=0$ i $y+z=1$.

3582*. Cilindrima $z=\ln x$ i $z=\ln y$ i ravnima $z=0$ i $x+y=2e$ $(x>1)$.

3583. Cilindrima $y=x+\sin x$, $y=x-\sin x$ i $z=\frac{(x+y)^2}{4}$ (parabolički cilindar čije su izvodnice paralelne pravoj $x-y=0$, $z=0$) i ravni $z=0$ $(0<x\leqslant <\pi,\ y>0)$.

Rješenja

3559. $186\frac{2}{3}$. **3560.** $\frac{ab}{6}\left(\frac{a^2}{p}+\frac{b^2}{q}\right)$.

3561. $\frac{abc}{6}$. **3562.** 12.

3563. $\frac{1}{6}$. **3564.** $78\frac{15}{32}$.

3565. $\frac{48}{5}\sqrt{6}$. **3566.** 16. **3567.** 45.

3568. $13\frac{1}{3}$. **3569.** $16\frac{1}{5}$.

3570. $ar^2\left(\frac{\pi}{4}-\frac{1}{3}\right)$. **3571.** 22π.

3572. $\frac{16}{3}R^3$. **3573.** $12\frac{4}{21}$.

3574. $\frac{4R^5}{15a^2}$. **3575.** 27. **3576.** $\frac{3}{8}$.

3577. $\frac{88}{105}$. **3578.** $\frac{1}{3}abc$.

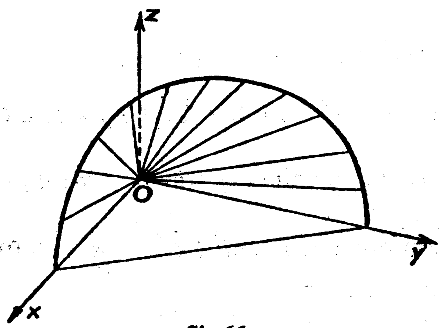
3579. $\frac{\pi a^3}{4}$. **3580.** $2\left(e^2-\frac{2e^3+1}{9}\right)$.

3581. $3e-8$.

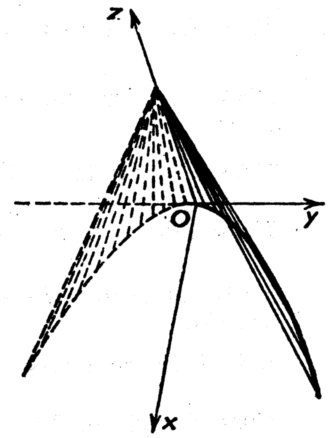
3582*. $4e-e^2-1$. Telo je simetrično u odnosu na ravan $y=x$.

3583. $2\left(\pi^2-\frac{35}{9}\right)$.

3584. Konusnom površinom $z^2=xy$ (sl. 66), cilindrom $\sqrt{x}+\sqrt{y}=1$ i ravni $z=0$.



Sl. 66



Sl. 67

3585. Konusnom površinom $4y^2=x(2-z)$ (parabolični konus, sl. 67) i ravnima $z=0$ i $x+z=2$.

3586. Površinom $z=\cos x\cdot\cos y$ i ravnima $x=0$, $y=0$, $z=0$ i $x+y=\frac{\pi}{2}$.

3587. Cilindrom $x^2+y^2=4$ i ravnima $z=0$ i $z=x+y+10$.

3588. Cilindrom $x^2+y^2=2x$ i ravnima $2x-z=0$ i $4x-z=0$.

3589. Cilindrom $x^2+y^2=R^2$, paraboloidom $Rz=2R^2+x^2+y^2$ i ravni $z=0$.

3590. Cilindrom $x^2+y^2=2ax$, paraboloidom $z=\frac{x^2+y^2}{a}$ i ravni $z=0$.

3591. Sferom $x^2+y^2+z^2=a^2$ i cilindrom $x^2+y^2=ax$ (Vivijanijev problem).

3592. Hiperboličkim paraboloidom $z=\frac{xy}{a}$, cilindrom $x^2+y^2=ax$ i ravni $z=0$ $(x>0,\ y>0)$.

3593. Cilindrima $x^2+y^2=x$ i $x^2+y^2=2x$, paraboloidom $z=x^2+y^2$ i ravnima $x+y=0$, $x-y=0$ i $z=0$.

3594. Cilindrima $x^2+y^2=2x$, $x^2+y^2=2y$ i ravnima $z=x+2y$ i $z=0$.

3595. Konusnom površinom $z^2=xy$ i cilindrom $(x^2+y^2)^2=2xy$ $(x>0,\ y>0,\ z\geqslant 0)$.

3596. Helikoidom („spiralne lestvice") $z=h\operatorname{arctg}\frac{y}{x}$, cilindrom $x^2+y^2=R^2$ i ravnima $x=0$ i $z=0$ $(x>0,\ y\geqslant 0)$.

Površina ravne oblasti

U zadacima 3597 — 3608 pomoću dvojnih integrala naći površine navedenih oblasti.

3597. Oblasti ograničene pravama $x=0$, $y=0$, $x+y=1$.

3598. Oblasti ograničene pravama $y=x$, $y=5x$, $x=1$.

3599. Oblasti ograničene elipsom $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3600. Oblasti ograničene parabolom $y^2=\frac{b^2}{a}x$ i pravom $y=\frac{b}{a}x$.

3601. Oblasti ograničene parabolama $y=\sqrt{x}$, $y=2\sqrt{x}$ i pravom $x=4$.

3602*. Oblasti ograničene krivom $(x^2+y^2)^2=2ax^3$

3603. Oblasti ograničene krivom $(x^2+y^2)^3=x^4+y^4$.

3604. Oblasti ograničene krivom $(x^2+y^2)^2=2a^2(x^2-y^2)$ (Bernulijeva lemniskata).

3605. Oblasti ograničene petljom krive $x^3+y^3=2xy$ koja leži u prvom kvadrantu.

3606. Oblasti ograničene petljom krive $(x+y)^3=xy$ koja leži u prvom kvadrantu.

3607. Oblasti ograničene petljom krive $(x+y)^5=x^2y^2$ koja leži u prvom kvadrantu.

Rješenja

3584. $\frac{1}{45}$. **3585.** $\frac{16}{9}$. **3586.** $\frac{\pi}{4}$.

3587. 40π. **3588.** 2π.

3589. $\frac{5}{2}\pi R^3$. **3590.** $\frac{3}{2}\pi a^3$.

3591. $\frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$. **3592.** $\frac{a^3}{24}$.

3593. $\frac{15}{8}\left(\frac{3\pi}{8}+1\right)$.

3594. $\frac{3}{2}\left(\frac{\pi}{2}-1\right)$. **3595.** $\frac{\pi\sqrt{2}}{24}$.

3596. $\frac{\pi^2R^2h}{16}$. **3597.** $\frac{1}{2}$.

3598. 2. **3599.** πab.

3600. $\frac{ab}{6}$. **3601.** $\frac{16}{3}$.

3602*. $\frac{5}{8}\pi a^2$; preći na polarne koordinate. **3603.** $\frac{3}{4}\pi$.

3604. $2a^2$. **3605.** $\frac{2}{3}$.

3606. $\frac{1}{60}$. **3607.** $\frac{1}{1260}$.

3608. Oblasti ograničene linijom

1) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{xy}{c^2}$; 2) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2+y^2}{25}$;

Površina površi

3626. Izračunati površinu onog dela ravni $6x+3y+2z=12$ koji leži u prvom oktantu.

3627. Izračuuati površinu onog dela površi $z^2=2xy$ koji se projektuje na pravougaonik u ravni $z=0$, ograničen pravama $x=0$, $y=0$, $x=3$, $y=6$.

3628. Naći površinu onog dela konusa $z^2=x^2+y^2$ koji leži između koordinatne ravni Oxy i ravni $z=\sqrt{2}\left(\frac{x}{2}+1\right)$.

U zadacima 3629 — 3639 naći površine nazn... delova datih površi.

3629. Dela $z^2=x^2+y^2$ isečenog cilindrom $z^2=2py$.

3630*. Dela $y^2+z^2=x^2$, koji leži unatar cilindra $x^2+y^2=R^2$.

3631. Dela $y^2+z^2=x^2$, koji isecaju cilindar $x^2-y^2=a^2$ i ravni $y=b$ i $y=-b$.

3632. Dela $z^2=4x$, koji isecaju cilindar $y^2=4x$ i ravan $x=1$.

3633. Dela $z=xy$, isečenog cilindrom $x^2+y^2=R^2$.

3634. Dela $2z=x^2+y^2$, isečenog cilindrom $x^2+y^2=1$.

3635. Dela $x^2+y^2+z^2=a^2$, isečenog cilindrom $x^2+y^2=R^2$ $(R<a)$.

3636. Dela $x^2+y^2+z^2=R^2$, isečenog cilindrom $x^2+y^2=Rx$.

2637. Dela $x^2+y^2+z^2=R^2$, koga iseca „lemniskatni" cilindar $(x^2+y^2)^2=R^2(x^2-y^2)$.

3638. Dela $z=\frac{x+y}{x^2+y^2}$ koji leži u prvom oktantu i isečen je cilindrima $x^2+y^2=1$ i $x^2+y^2=4$.

3639. Dela $(x\cos\alpha+y\sin\alpha)^2+z^2=a^2$, koji leži u prvom oktantu $\left(\alpha<\frac{\pi}{2}\right)$.

3640*. Izračunati površinu dela zemljine kugle (smatrajući zemlju loptom poluprečnika $R\approx 6400\,km$) ograničenog meridijanima $\varphi=30°$ i $\varphi=60°$, i uporednicima $\theta=45°$ i $\theta=60°$.

3641. Izračunati ukupnu površinu tela ograničenog sferom $x^2+y^2+z^2=3a^2$ i paraboloidom $x^2+y^2=2az$ $(z\geqslant 0)$.

3642. Ose dva istovetna cilindra poluprečnika R seku se pod pravim uglom; naći površinu onog dela jednog cilindra koji leži u drugom cilindru.

Rješenja

3608*. 1) $\frac{a^2b^2}{2c^2}$; 2) $\frac{39}{25}\pi$; iskoristiti tvrđenje formulisano u zad. 3541.

3626. 14. **3627.** 36.

3628. 8π. **3269.** $2\sqrt{2}\pi p^2$

3630*. $2\pi R^2$. Projicirati površinu na ravan Oyz.

3631. $8\sqrt{2}ab$. **3632.** $\frac{16}{3}(\sqrt{8}-1)$.

3633. $\frac{2\pi}{3}\{(1+R^2)^{\frac{3}{2}}-1\}$.

3634. $\frac{2\pi}{3}(\sqrt{8}-1)$.

3635. $4\pi a(a-\sqrt{a^2-R^2})$.

3636. $2R^2(\pi-2)$.

3637. $2R^2(\pi+4-4\sqrt{2})$.

3638. $\frac{\pi}{4}\{3-\sqrt{2}-\sqrt{3}-\frac{\sqrt{2}}{2}\ln 2+\sqrt{2}\ln(\sqrt{3}+\sqrt{2})\}$.

3639. $\frac{2a^2}{\sin 2\alpha}$.

3640*. $\frac{\pi R^2}{12}(\sqrt{3}-\sqrt{2})\approx 3{,}42\cdot 10^8\,km^2$. Preći na sferne koordinate.

3641. $\frac{16}{3}\pi a^2$. **3642.** $8R^2$.

Primjena trostrukog integrala

a) Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi

$$V = \iiint\limits_{\Omega} dx\,dy\,dz$$

b) Težište $T(x_T, y_T, z_T)$ trodimenzionalnog homogenog tijela ograničenog oblašću Ω tražimo po formulama

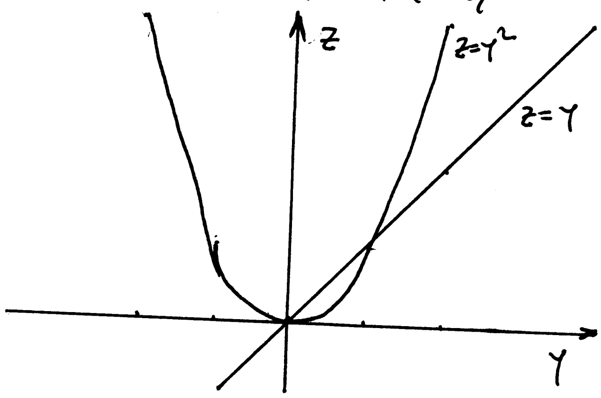
$$x_T = \frac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz, \qquad y_T = \frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz$$

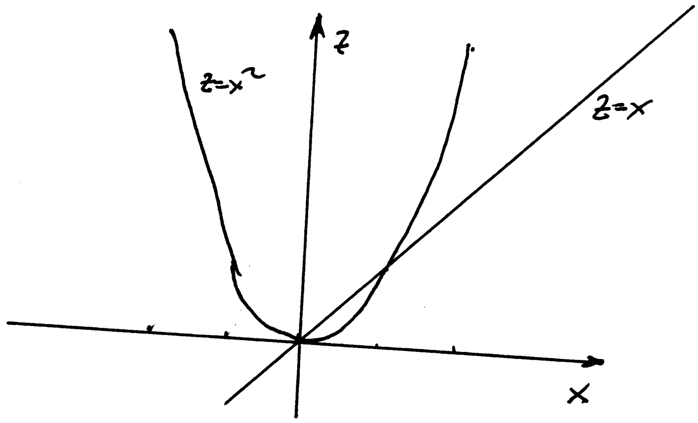
$$z_T = \frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz$$

Homogeno tijelo je tijelo kojem je masa jednako raspoređena u svim njegovim dijelovima

(#) Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

Rj.: Pogledajmo kako izgleda presjek datih površina sa yOz i xOz ravni





Na osnovu ove dvije slike pokušajte skicirati tijelo u prostoru!

$$V=\iiint_{\Omega} dx\,dy\,dz = \iint_D dx\,dy \int_{x^2+y^2}^{x+y} dz = \iint_D \left(x+y-(x^2+y^2)\right) dx\,dy \overset{(*)}{=}$$

gdje je D ortogonalna projekcija datog tijela na xOy ravan.

Projekciju presjeka tijela određujemo na sljedeći način

$$\begin{array}{l} z=x+y \\ z=x^2+y^2 \\ \hline x+y=x^2+y^2 \end{array} \Rightarrow x^2-x+y^2-y=0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}=\frac{1}{2}$$

$$D:\ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$$

Ako uvedemo polarne koordinate $x=\frac{1}{2}+r\cos\varphi$, $y=\frac{1}{2}+r\sin\varphi$, $dx\,dy = r\,dr\,d\varphi$

$D \xrightarrow{\text{transformiramo}} D'$

$$D' : \begin{cases} 0 \leq r \leq \frac{1}{\sqrt{2}} \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$\overset{(*)}{=} \iint_D (-1)\left(x^2 - x + y^2 - y\right) dx\,dy = (-1) \iint_D \left(\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 - \frac{1}{2}\right) dx\,dy =$$

Primijetimo da je $x - \frac{1}{2} = r\cos\varphi$

$y - \frac{1}{2} = r\sin\varphi$

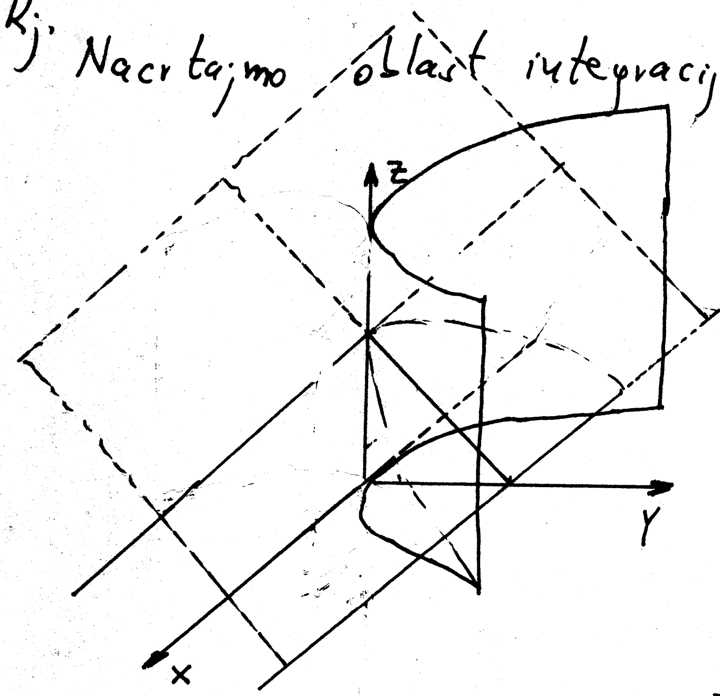
$$= (-1) \iint_{D'} \left(r^2 - \frac{1}{2}\right) r\,dr\,d\varphi = (-1) \int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \left(r^3 - \frac{1}{2}r\right) dr = \ldots = \frac{\pi}{8}$$

traženo rješenje

Ⓗ Izračunati zapreminu tijela koje je ograničeno cilindrom $Y=2x^2$ i ravnima $Y+z=8$, $z=0$.

Rj. Nacrtajmo oblast integracije

$$\Omega: \begin{cases} Y=2x^2 \\ Y+z=8 \\ z=0 \end{cases}$$



Ravan $Y+z=8$ siječe cilindar

Napravimo projekciju oblasti Ω na xOY ravan.

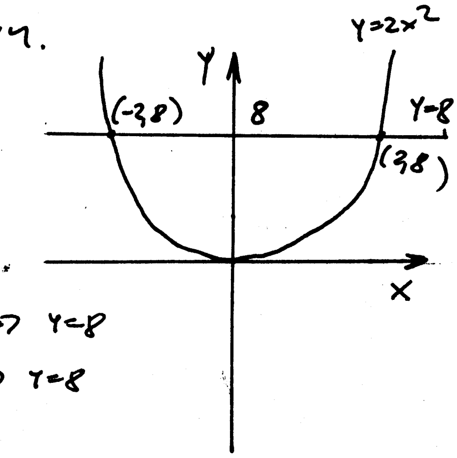
Nađimo presjek krive $Y=2x^2$ i prave $Y=8$.

$$\begin{aligned} Y&=2x^2 \\ Y&=8 \\ \hline x^2&=4 \end{aligned}$$

$x_1=-2$, $x_2=2$

$x_1=-2 \Rightarrow Y=8$

$x_2=2 \Rightarrow Y=8$



$$\Omega: \begin{cases} -2 \le x \le 2 \\ 2x^2 \le Y \le 8 \\ 0 \le z \le 8-Y \end{cases}$$

$$V = \iiint_{\Omega} dx\,dY\,dz$$

$$V = \iiint_{\Omega} dx\,dY\,dz = \int_{-2}^{2} dx \int_{2x^2}^{8} dY \int_{0}^{8-Y} dz = \int_{-2}^{2} dx \int_{2x^2}^{8} z\Big|_0^{8-Y} dY = \int_{-2}^{2} dx \int_{2x^2}^{8} (8-Y)\,dY =$$

$$= \int_{-2}^{2} \left(8Y\Big|_{2x^2}^{8} - \frac{1}{2}Y^2\Big|_{2x^2}^{8}\right) dx = \int_{-2}^{2} \left[8(8-2x^2) - \frac{1}{2}(8^2-4x^4)\right] dx =$$

$$= \int_{-2}^{2} (64-16x^2-32+2x^4)\,dx = \int_{-2}^{2} (2x^4-16x^2+32)\,dx =$$

$$= 2\cdot\frac{1}{5}x^5\Big|_{-2}^{2} - 16\cdot\frac{1}{3}x^3\Big|_{-2}^{2} + 32x\Big|_{-2}^{2} = \frac{2}{5}\cdot 64 - \frac{16}{3}\cdot 16 + 32\cdot 4 =$$

$$= \frac{384 - 1280 + 1920}{15} = \frac{1024}{15}$$

(#) Izračunati zapreminu tijela ograničenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

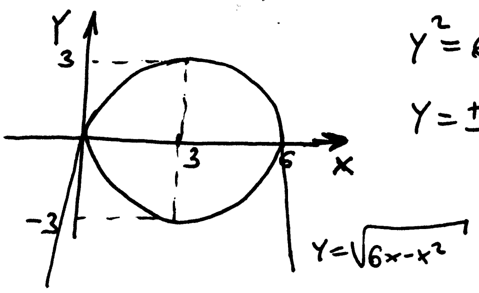
Rj. $V=\iiint\limits_{\Omega} dx\,dy\,dz$

$x^2+y^2=6x$

$x^2-2\cdot x\cdot 3+3^2-3^2+y^2=0$

$(x-3)^2+y^2=3^2$

projekcija valjka na xOy ravan izgleda



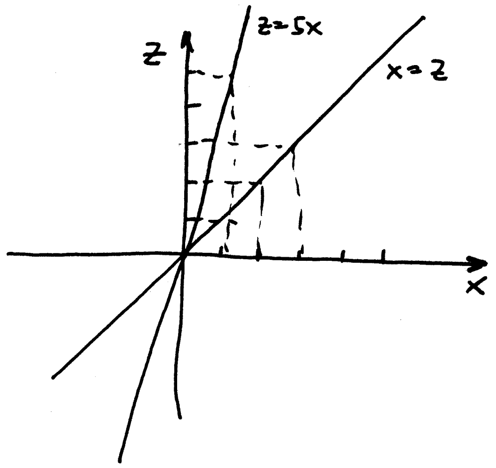
$y^2=6x-x^2$

$y=\pm\sqrt{6x-x^2}$

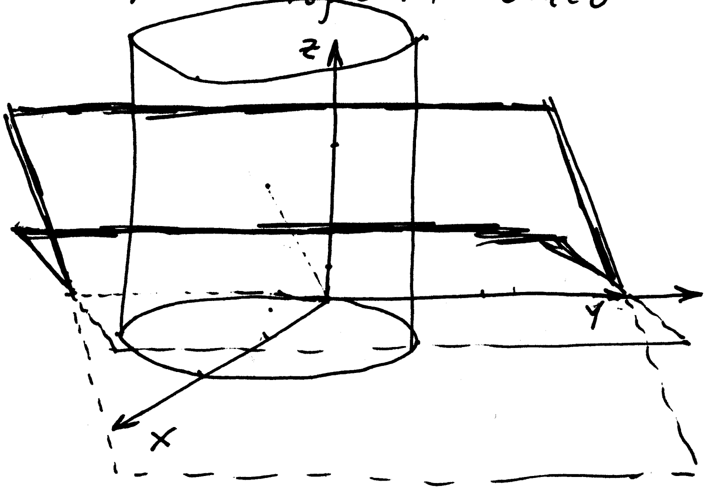
$x-z=0$ $5x-z=0$

$x=z$ $z=5x$

projekcije ravni $x-z=0$ i $5x-z=0$ na xOz ravan izgleda



Skica ovih figura u prostoru bi otprilike izgledala ovako



valjak presječen sa dvije ravni

na klasičan način

$$\Omega:\begin{cases}0<x<6\\ 0<y<\sqrt{6x-x^2}=\sqrt{9-(x-3)^2}\\ x\le z\le 5x\end{cases}$$

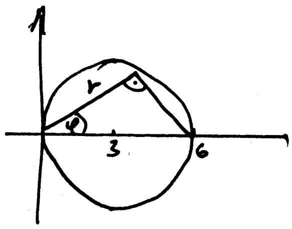
uvedimo cilindrične koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dx\,dy=r\,dr\,d\varphi$



$\cos\varphi=\frac{r}{6}$

Primjetimo da je oblast Ω simetrična u odnosu na xOz ravan

$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz$

$$\Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 6\cos\varphi\\ r\cos\varphi<z<5r\cos\varphi\end{cases}$$

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=2\int_0^{\pi/2}d\varphi\int_0^{6\cos\varphi}r\,dr\underbrace{\int_{r\cos\varphi}^{5r\cos\varphi}dz}_{=4r\cos\varphi}=8\int_0^{\pi/2}\cos\varphi\,d\varphi\int_0^{6\cos\varphi}r^2\,dr=8\int_0^{\pi/2}\frac{1}{3}r^3\Big|_0^{6\cos\varphi}\cos\varphi\,d\varphi$$

$$=8\cdot 6^3/3\int_0^{\pi/2}\cos^4\varphi\,d\varphi=576\int_0^{\pi/2}\left(\frac{1}{2}(1+\cos 2\varphi)\right)^2d\varphi=144\int_0^{\pi/2}(1+2\cos 2\varphi+\cos^2 2\varphi)\,d\varphi\overset{\text{za vježbu}}{=}\dots=108\pi$$

tražena zapremina

Izračunati zapreminu tijela ograničenog ravninom XOY, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint_{\Omega} dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

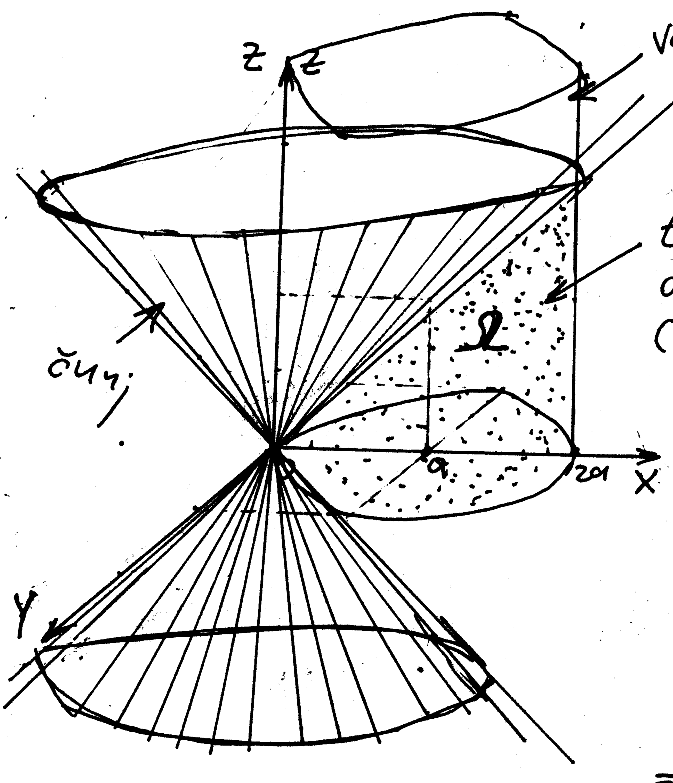
$$x^2-2ax+y^2=0$$

$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$

$$(x-a)^2+y^2=a^2$$

valjak u presjeku sa XOY ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa XOY ravni je tačka, a u presjeku sa YOZ ili sa XOZ su po dvije prave



tražimo zapreminu ovog tijela (na slici smo pretpostavili da je $a>0$)

Oblast Ω je najlakše projicirati na XOY ravan.

Uvodimo cilindrične koordinate

$$x=a+r\cos\varphi$$
$$y=r\sin\varphi$$
$$z=z$$

$$\Omega':\begin{cases} dx\,dy\,dz=r\,dr\,d\varphi\,dz \\ 0\le r\le a \\ 0\le\varphi\le 2\pi\end{cases}$$

$$x^2+y^2=z^2$$

$$z=\pm\sqrt{x^2+y^2} \quad \text{čunj}$$

$$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$$
$$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$$
$$=a^2+2ar\cos\varphi+r^2$$

$$V=\iiint_{\Omega} dx\,dy\,dz=\iiint_{\Omega'} r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}} r\,dz = \dots$$ ovo je teško izračunati

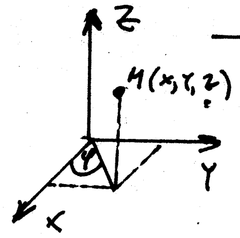
Pokušajmo uvesti drugačije smjene.

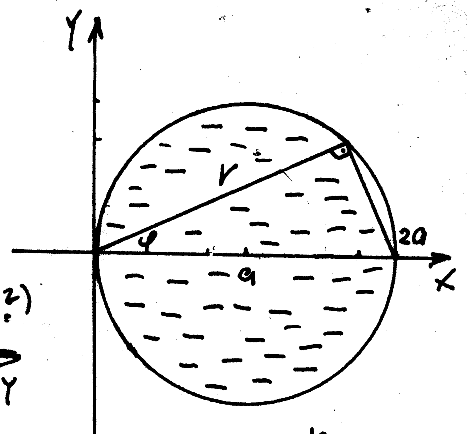
$x = r\cos\varphi$ $\qquad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$y = r\sin\varphi$

$z = z$ $\qquad x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$

$$\Omega'' : \begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$





$\cos\varphi = \frac{r}{2a}$

$r = 2a\cos\varphi$

$$V = \iiint_{\Omega} dx\,dy\,dz = \iiint_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^{r} r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(rz \Big|_0^r \right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^{2a\cos\varphi} \right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\,(1-\sin^2\varphi)\,d\varphi = \left| \begin{matrix} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{matrix} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,dt = t\Big|_{-1}^{1} - \frac{1}{3}t^3\Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$V = \frac{32}{9} a^3$ tražena zapremina

II način: $V = \iint_S f(x,y)\,dx\,dy$ uvedimo smjene $\quad x = r\cos\varphi$

$y = r\sin\varphi$

$0 \le \varphi \le 2a\cos\varphi$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr \qquad \text{ZAVRŠITI ZA VJEŽBU}$$

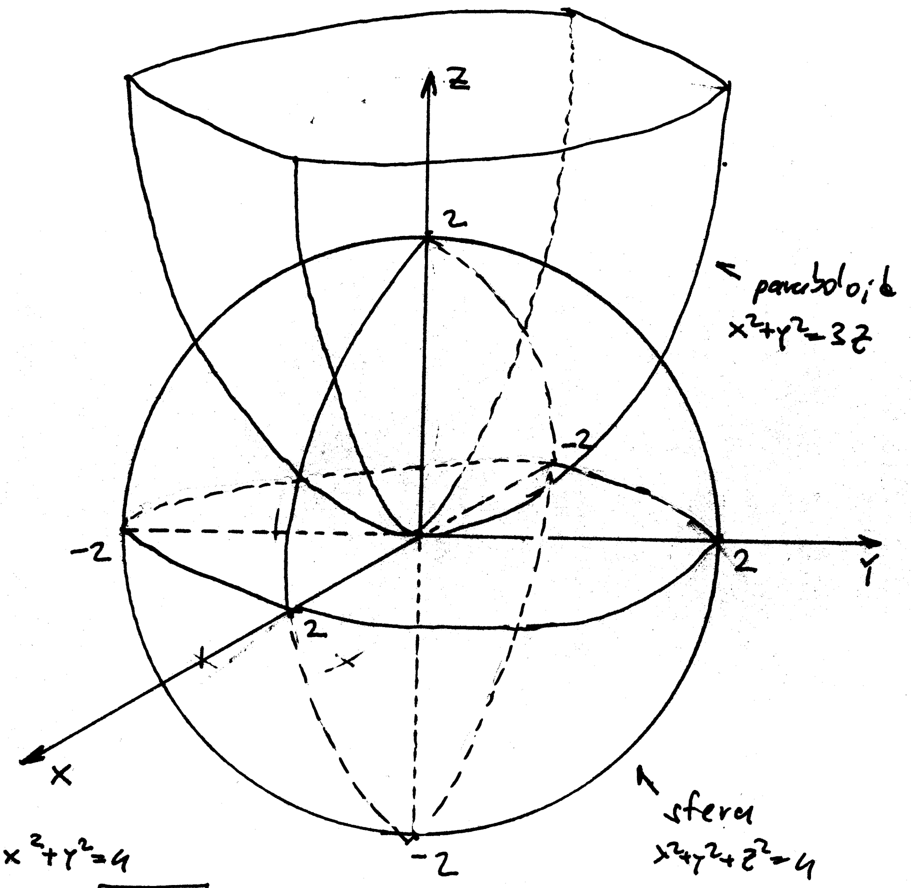
$-\frac{\pi}{2} \le r \le \frac{\pi}{2}$

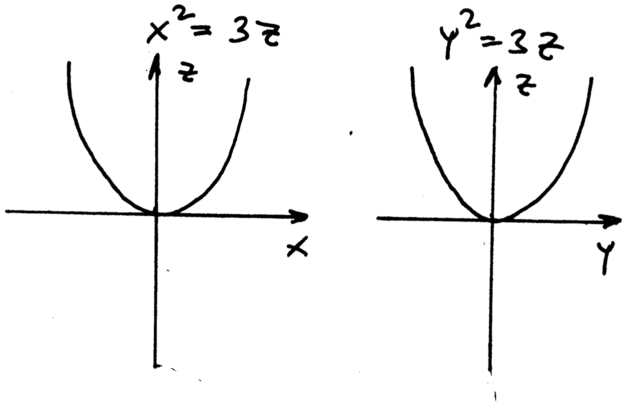
(#) Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u $(0,0,0)$ poluprečnika 2

$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela





$x^2+y^2=4$

$x=\pm\sqrt{4-y^2}$

$y=\pm\sqrt{4-x^2}$

$$V=\iiint_{\Omega} dx\,dy\,dz$$

Primetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz.

Prema tome

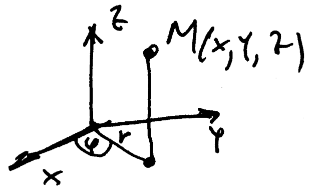
$$V=4\iiint_{\Omega_1} dx\,dy\,dz \quad \text{gdje je}$$

Ω_1 oblast u presjeku dva tijela u prvom oktantu

$$\Omega_1:\begin{cases} 0\le x\le 2 \\ 0\le y\le \sqrt{4-x^2} \\ 0\le z\le \frac{1}{3}(x^2+y^2) \end{cases}$$

$$V=4\int_0^2 dx\int_0^{\sqrt{4-x^2}} dy\int_0^{\frac{1}{3}(x^2+y^2)} dz = 4\int_0^2 dx\int_0^{\sqrt{4-x^2}} \frac{1}{3}(x^2-y^2)\,dy$$

$$=\frac{4}{3}\int_0^2 \left(x^2 y\Big|_0^{\sqrt{4-x^2}} + \frac{1}{3}y^3\Big|_0^{\sqrt{4-x^2}}\right)dx = \underset{\text{komplikovano}}{\ldots} = \frac{8\pi}{3}$$



II način:

Uvedimo cilindrične koordinate

$x=r\cos\varphi$
$y=r\sin\varphi$
$z=z$

$dx\,dy\,dz=r\,dr\,d\varphi\,dz$

Oblast $\Omega_1 \xrightarrow{\text{transformise}} \Omega_1' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2} dr \int_0^{\frac{1}{3}r^2} r\, dz = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2} r \cdot \frac{1}{3} r^2\, dr = \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^2 d\varphi = \frac{1}{3} \cdot 16 \cdot \frac{\pi}{2} = \frac{8\pi}{3}$$

$V = \frac{8\pi}{3}$ tražena zapremina

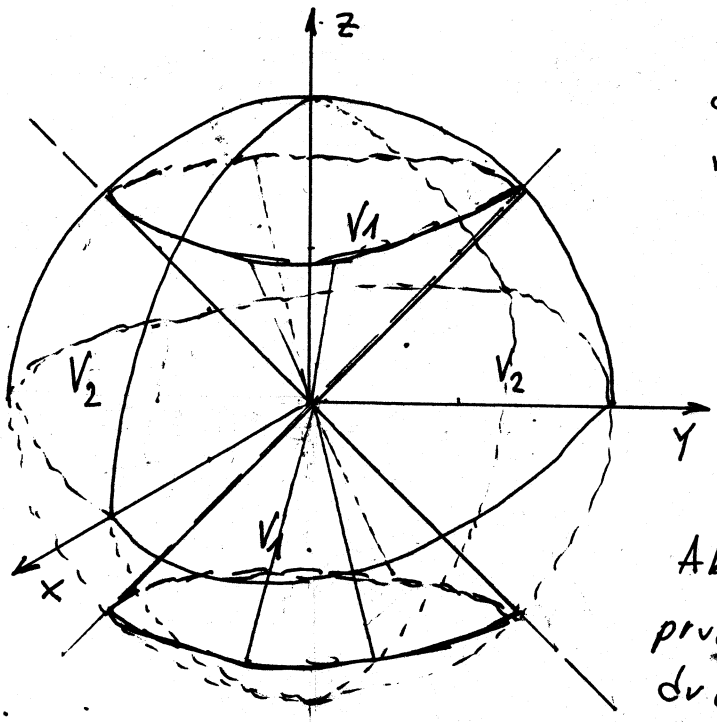
Izračunati zapreminu tijela koje je ograničeno površima $z^2=x^2+y^2$, $x^2+y^2+z^2=4$.

Rj.

$x^2+y^2+z^2=4$ je kugla sa centrom u $(0,0,0)$ poluprečnika $r=2$

$z^2=x^2+y^2$ je konus

Skicirajmo ove dvije figure u prostoru.



Presjek konusa i kugle daje dva tijela za koje možemo računati zapreminu: prvo tijelo je određeno u presjeku unutrašnjosti konusa i kugle, a drugo tijelo je određeno dijelom lopte van konusa. Ako sa V_1 označimo zapreminu prvog, a sa V_2 zapreminu drugog tijela, imamo da je

$$V=V_1+V_2=\frac{4}{3}r^3\pi \quad (\text{zapremina kugle})$$

Kako je $r=2 \Rightarrow V=\frac{4}{3}\cdot 8\pi=\frac{32\pi}{3}$.

$V=\iiint\limits_{\Omega} dx\,dy\,dz$ – zapremina tijela ograničenog sa oblašću Ω

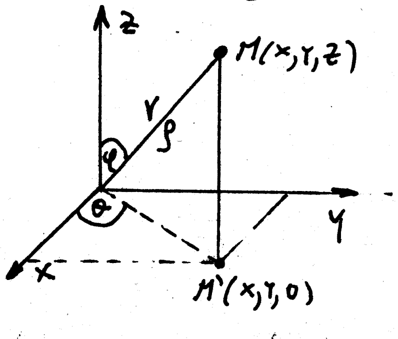
Uvedimo sferne koordinate

$x=\rho\sin\varphi\cos\theta$

$y=\rho\sin\varphi\sin\theta$

$z=\rho\cos\varphi$

$dx\,dy\,dz=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$



$z^2=x^2+y^2$

$\rho^2\cos^2\varphi=\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta=$

$=\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)=\rho^2\sin^2\varphi$

$\Rightarrow \cos^2\varphi=\sin^2\varphi \quad /:\sin^2\varphi$

$\operatorname{tg}^2\varphi=1 \Rightarrow \operatorname{tg}\varphi=\pm 1$

$x^2+y^2+z^2=4 \Rightarrow \rho^2=4 \Rightarrow \rho=\pm 2$ tj. $\rho=2$ udaljenost tačke

$$\Omega:\begin{cases} z^2=x^2+y^2 \\ x^2+y^2+z^2=4 \end{cases} \xrightarrow{\text{transformacije}} \Omega':\begin{cases} \operatorname{tg}\varphi=\pm 1 \\ \rho=2 \end{cases}$$

Odredimo granice za drugo tijelo $\Omega'_{V_2}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \end{cases}$

$$V_2 = \iiint\limits_{\Omega'_{V_2}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 d\rho =$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^2 = 2\pi\left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} =$$

$$= 2\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 2\pi\sqrt{2} \cdot \frac{8}{3} = \frac{16\pi\sqrt{2}}{3}$$ traženo rješenje

Zapreminu V_1 sad možemo odrediti na dva načina

I način:

$$V = V_1 + V_2 = \frac{32\pi}{3} \Rightarrow V_1 = \frac{32\pi}{3} - V_2 = \frac{32\pi}{3} - \frac{16\pi\sqrt{2}}{3}$$

$$V_1 = \frac{16\pi}{3}(2 - \sqrt{2})$$ traženo rješenje

II način:

Ako uzmemo u obzir simetričnost date oblasti Ω' u odnosu na xOy-ravan, možemo računati polovinu zapremine V_1 za $z \ge 0$ i tada bi trebalo odabrati sljedeće granice $\Omega'_{V_1}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases}$

$$V_1 = \iiint\limits_{\Omega'_{V_1}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\frac{1}{2}V_1 = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^2 \rho^2 d\rho = 2\pi(-\cos\varphi)\Big|_0^{\pi/4} \cdot \frac{\rho^3}{3}\Big|_0^2 =$$

$$= 2\pi\left(1 - \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3}$$

$$\Rightarrow V_1 = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 4\pi \cdot \frac{2 - \sqrt{2}}{2} \cdot \frac{8}{3} = \frac{16\pi}{3}(2 - \sqrt{2}).$$

(#) Izračunati zapreminu tijela koje je određeno oblašću $\Omega: |x+y+z|+|x-y+z|+|x+y-z|=1$.

Rj. $V=\iiint\limits_{\Omega} dx\,dy\,dz$

uvedimo smjenu
$$u=x+y+z$$
$$v=x-y+z$$
$$w=x+y-z$$

$dx\,dy\,dz = J\,du\,dv\,dw$ ← Jakobijan

$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad J^{-1}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} I_v+II_v \\ \hline\hline II_v+III_v \end{matrix}$$

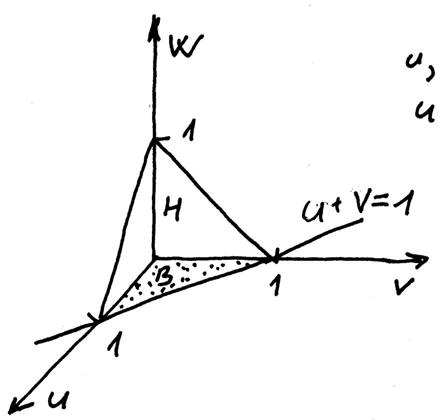
$$=\begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (-1)\cdot\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4)=4 \Rightarrow$$

$$\Rightarrow J=\frac{1}{4}$$

pa je
$$dx\,dy\,dz=\frac{1}{4}\,du\,dv\,dw$$

$\Omega': |u|+|v|+|w|=1$

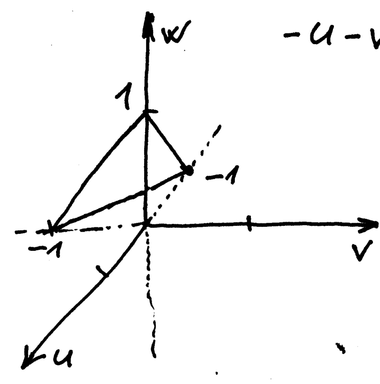
$$V=\iiint\limits_{\Omega'}\frac{1}{4}\,du\,dv\,dw$$



$u,v,w>0$
$u+v+w=1$

pored ovoga imamo još 7 slučajeva

npr. $u,v<0 \quad w>0$
$$-u-v+w=1$$



Vidimo da je dovoljno oblast integrirati u 1. oktantu jer imamo simetričnu oblast po svim oktantima.

$$V=8\cdot\frac{1}{4}\iiint\limits_{\Omega''}du\,dv\,dw =$$

$$=2\int_0^1 du\int_0^{1-u}dv\int_0^{1-u-v}dw = 2\int_0^1 du\int_0^{1-u} w\Big|_0^{1-u-v}dv =$$

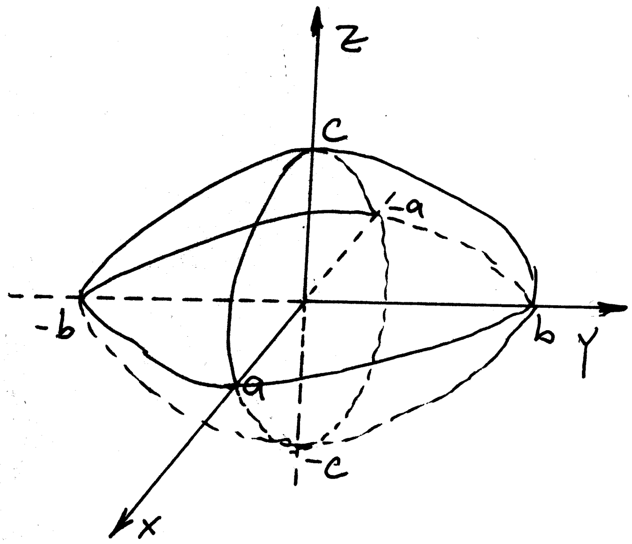
$u+v=1$
$v=1-u$
$u+v+w=1$
$w=1-u-v$

$$=2\int_0^1 du\int_0^{1-u}(1-u-v)\,dv = 2\int_0^1\left(v\Big|_0^{1-u}-uv\Big|_0^{1-u}-\frac{1}{2}v^2\Big|_0^{1-u}\right)du = \ldots = 2\cdot\frac{1}{6}=\frac{1}{3}$$

Na drugi način: $V_1=\frac{B\cdot H}{3}=\frac{\frac{1\cdot 1}{2}\cdot H}{3}=\frac{\frac{1}{2}\cdot 1}{3}=\frac{1}{6}$, $\quad V=2\cdot\frac{1}{6}=\frac{1}{3}$ zapremina tijela

Izračunati zapreminu elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Rj.



$$V=\iiint_S dx\,dy\,dz$$

$$S: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

smjena: uopštene sferne koordinate

$x = a r \sin\varphi\cos\theta \qquad 0\le r\le 1$

$y = b r \sin\varphi\sin\theta \qquad 0\le\varphi\le\pi$

$z = c r\cos\varphi \qquad 0\le\theta\le 2\pi$

$$dx\,dy\,dz = J\,dr\,d\varphi\,d\theta$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix} a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}$$

$$= abc\begin{vmatrix}\text{ista determinanta}\\ \text{kao kod standardnih}\\ \text{sfernih koordinata}\end{vmatrix} = abc\,r^2\sin\varphi$$

$$V=\int_0^{\pi}d\varphi\int_0^{1}dr\int_0^{2\pi}abc\,r^2\sin\varphi\,d\theta=\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr\int_0^{2\pi}abc\,d\theta=$$

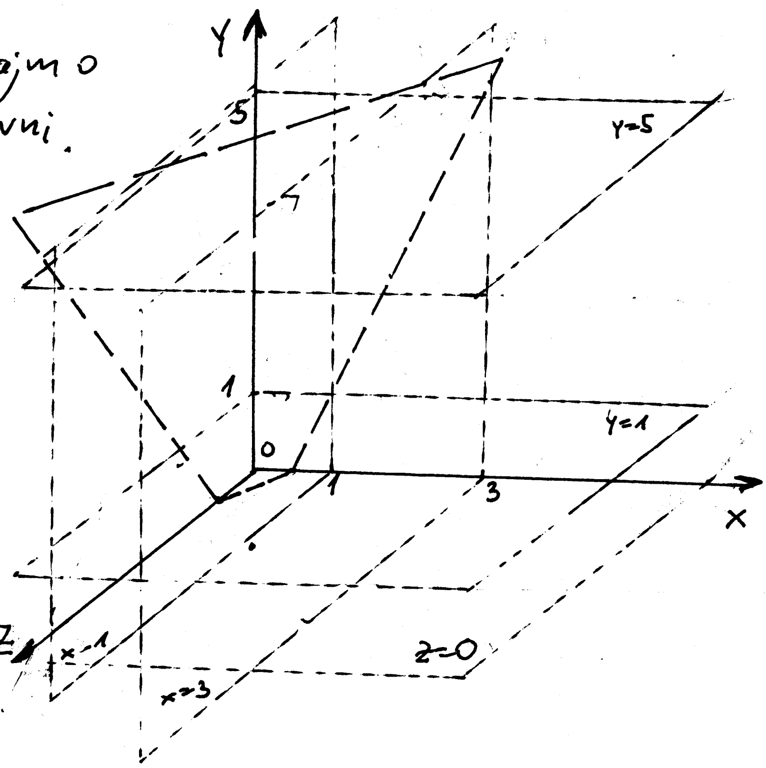
$$=abc\,\theta\Big|_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr=2\pi abc\int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^{1}d\varphi=$$

$$=\frac{2}{3}\pi abc\int_0^{\pi}\sin\varphi\,d\varphi=\frac{2}{3}\pi abc\left(-\cos\varphi\Big|_0^{\pi}\right)=\frac{2}{3}\pi abc\,(1+1)=\frac{4}{3}\pi abc$$

q.e.d.

Naći zapreminu tijela ograničenog ravnima $x=1$, $x=3$, $y=1$, $y=5$, $2x-y+z-1=0$, $z=0$.

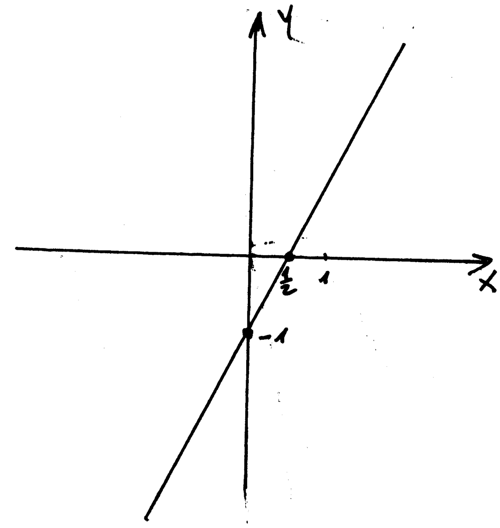
Rj. Nacrtajmo ove ravni.



$$2x-y+z-1=0$$

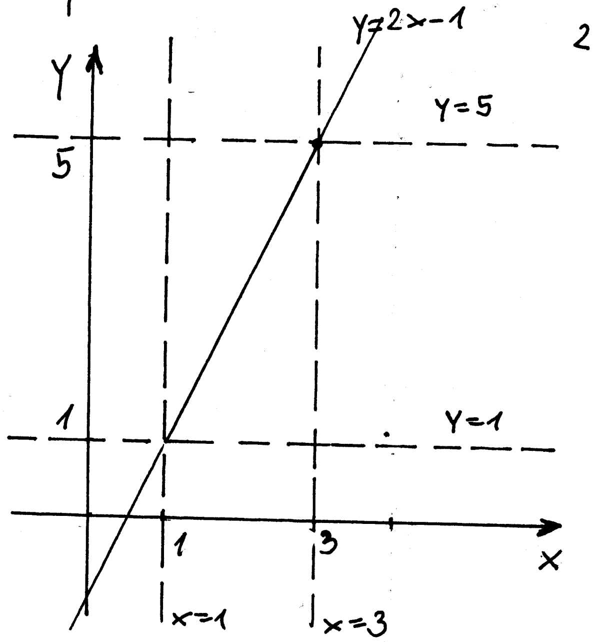
$$z=-2x+y+1$$

projekcija ove ravni na xOy ravan



$$y=2x-1$$

Slika u prostoru je komplikovana i sa nje ne možemo pročitati granice. Nacrtajmo projekcije ovih ravni na xOy ravan.



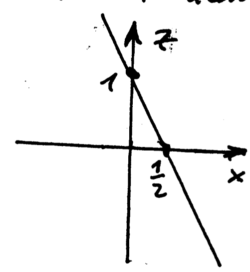
$$2x-y-1=0$$

$$y=2x-1$$

$$x=3 \Rightarrow y=5$$

$$x=1 \Rightarrow y=1$$

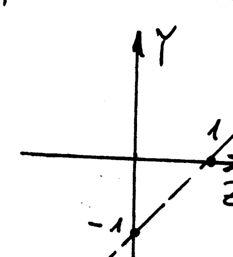
na xOz ravan



$$2x+z-1=0$$

$$z=-2x+1$$

na yOz ravan



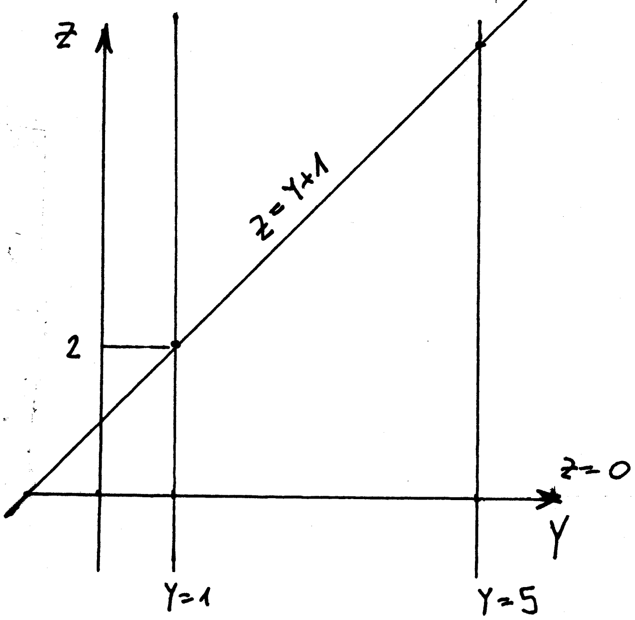
$$-y+z-1=0$$

$$y=z-1$$

Sad na osnovu slike u prostoru i projekcija na ravni možemo pročitati granice za tijelo

$$\Omega : \begin{cases} 1 \le x \le 3 \\ 2x-1 \le y \le 5 \\ 0 \le z \le -2x+y+1 \end{cases}$$

Da su napisane granice ispravne provjerimo projekcijom ravni na yOz ravan.



$$-y+z-1=0$$

$$z=y+1$$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz =$$

$$=\int_1^3 dx\int_{2x-1}^{5} dy\underbrace{\int_0^{-2x+y+1} dz}_{z\Big|_0^{-2x+y+1}} =$$

$$=\int_1^3 dx\int_{2x-1}^{5}(-2x+y+1)\,dy=\int_1^3\left((-2x)\cdot y\Big|_{2x-1}^{5}+\frac{1}{2}y^2\Big|_{2x-1}^{5}+y\Big|_{2x-1}^{5}\right)dx=$$

$$=\int_1^3\left((-2x)(5-(2x-1))+\frac{1}{2}\left(5^2-(2x-1)^2\right)+5-(2x-1)\right)dx=$$

$$=\int_1^3\left((-2x)(6-2x)+\frac{1}{2}\left(25-(4x^2-4x+1)\right)+6-2x\right)dx=$$

$$=\int_1^3\Big(\underline{\underline{-12x}}+4x^2+\overbrace{\frac{1}{2}(-4x^2+4x+24)}^{-2x^2+\underline{\underline{2x}}+12}+6\underline{\underline{-2x}}\Big)dx=\int_1^3(2x^2-12x+18)\,dx$$

$$=\frac{2}{3}x^3\Big|_1^3-\frac{12}{2}x^2\Big|_1^3+18x\Big|_1^3=\frac{2}{3}\cdot 26-\overset{48}{6\cdot 8}+\overset{36}{18\cdot 2}=\frac{52}{3}-12=\frac{16}{3}$$

Zapremina tijela ograničenog spomenutim ravnima iznosi $\frac{16}{3}$.

#) Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$, $a>0$ u I oktantu.

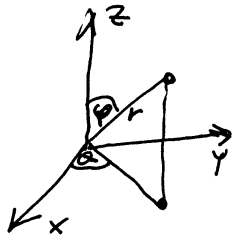
Rj.

Zapremina tijela ograničenog sa oblasti Ω se računa po formuli $V=\iiint\limits_{\Omega} dx\,dy\,dz.$

Datu površ $(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$ ne možemo skicirati.

Uvedimo sferne koordinate

$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$



$$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$\Omega \xrightarrow{\text{transformacije}} \Omega'$$

pa pokušajmo naći granice na osnovu date formule.

$$x^2+y^2+z^2=\underline{r^2\sin^2\varphi\cos^2\theta}+\underline{r^2\sin^2\varphi\sin^2\theta}+r^2\cos^2\varphi=r^2\sin^2\varphi+r^2\cos^2\varphi=r^2$$

$$(x^2+y^2+z^2)^3=(r^2)^3=r^6$$
$$z^2=r^2\cos^2\varphi$$
$$x^2+y^2=r^2\sin^2\varphi$$

$$(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$$

sad postaje $r^6=\frac{a^6r^2\cos^2\varphi}{r^2\sin^2\varphi}$

tj. $r^6=a^6\,\mathrm{ctg}^2\varphi$

$$r=\sqrt[6]{a^6\,\mathrm{ctg}^2\varphi}$$
$$r=a\sqrt[3]{\mathrm{ctg}\varphi}.$$

Na osnovu ove formule i znanja da je tijelo u I oktantu možemo zaključiti da je

$$\Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le a\sqrt[3]{\mathrm{ctg}\varphi}\\ 0\le\theta\le\frac{\pi}{2}\end{cases}$$

$$V=\iiint\limits_{\Omega'} r^2\sin\varphi\,dr\,d\varphi\,d\theta=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\varphi\,d\varphi\int_0^{a\sqrt[3]{\mathrm{ctg}\varphi}}r^2\,dr=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\varphi\,\frac{r^3}{3}\Big|_0^{a\sqrt[3]{\mathrm{ctg}\varphi}}d\varphi$$

$$=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\frac{a^3}{3}\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}\,d\varphi=\frac{a^3}{3}\int_0^{\pi/2}d\theta\int_0^{\pi/2}\cos\varphi\,d\varphi=\frac{a^3}{3}\cdot\theta\Big|_0^{\frac{\pi}{2}}\cdot\sin\varphi\Big|_0^{\frac{\pi}{2}}=\frac{a^3\pi}{6}$$ tražena zapr.

Računanje težišta tijela

U slijedećim zadacima izračunajte koordinate težišta tijela (oblasti) Ω ograničenog datim površima!

1. $\Omega: z^2 = xy \wedge x = 5 \wedge y = 5 \wedge z = 0$.

Rješenje: najprije ćemo izračunati zapreminu date oblasti Ω. Očito je $0 \le z \le \sqrt{xy}$, a iz $z^2 = xy$ slijedi $xy \ge 0$, pa je $0 \le x \le 5 \wedge 0 \le y \le 5$. Zato je

$$V = \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \int_0^5 \sqrt{x}dx \int_0^5 \sqrt{y}dy = \left(\int_0^5 \sqrt{x}dx \right)^2 = \frac{500}{9}.$$

Dalje imamo da je

$$\overline{x} = \frac{9}{500} \int_0^5 x\,dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \frac{9}{500} \int_0^5 x\sqrt{x}\,dx \int_0^5 \sqrt{y}dy = \frac{9}{500} \int_0^5 x^{\frac{3}{2}}\,dx \int_0^5 y^{\frac{1}{2}} dy = \ldots = 3.$$

Očigledno je $\overline{x} = \overline{y}$. Najzad,

$$\overline{z} = \frac{9}{500} \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} z\,dz = \frac{9}{500} \cdot \frac{1}{2} \int_0^5 xdx \int_0^5 ydy = \frac{9}{1000} \left[\left. \frac{x^2}{2} \right|_0^5 \right]^2 = \frac{9}{1000} \cdot \frac{25}{2} \cdot \frac{25}{2} = \frac{45}{32}.$$

Dakle, težište ima koordinate $T\left(3, 3, \frac{45}{32}\right)$.

2. $\Omega: z = 3 - x^2 - y^2, z = 0.$

Rješenje: Uvešćemo cilindrične koordinate. Tada se Ω preslikava u oblast: $\Omega': z = 3 - \rho^2, z = 0.$

U presjeku ove dvije površi se dobija kružnica $\rho^2 = 3 \Rightarrow \rho = \sqrt{3}.$ Zato je $0 \le \varphi \le 2\pi,\ 0 \le \rho \le \sqrt{3},\ 0 \le z \le 3 - \rho^2$. Odatle slijedi:

$$V = \iiint_{\Omega'} \rho d\varphi\, d\rho\, dz = \int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} dz = 2\pi \int_0^{\sqrt{3}} \rho\left(3 - \rho^2\right) d\rho = 2\pi \int_0^{\sqrt{3}} \left(3\rho - \rho^3\right) d\rho =$$

$$= 2\pi \left. \left(3\frac{\rho^2}{2} - \frac{\rho^4}{4} \right) \right|_0^{\sqrt{3}} = 2\pi \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{9\pi}{2}.$$

Sada možemo izračunati koordinate težišta tijela:

$$\overline{x} = \frac{1}{V} \iiint_{\Omega} x\,dx\,dy\,dz = \frac{2}{9\pi} \iiint_{\Omega'} \rho\cos\varphi \cdot \rho\, d\varphi\, d\rho\, dz = \frac{2}{9\pi} \int_0^{2\pi} \cos\varphi\, d\varphi \int_0^{\sqrt{3}} \rho^2 d\rho \int_0^{3-\rho^2} dz = 0,$$

jer je

$\int_0^{2\pi} \cos\varphi\, d\varphi = 0.$ Na isti način dobijamo da je $\overline{y} = 0.$ I najzad,

$$\bar{z} = \frac{1}{V}\iiint\limits_{\Omega'} \rho z \, d\varphi \, d\rho \, dz = \frac{2}{9\pi}\int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} z \, dz = \frac{4\pi}{9\pi}\int_0^{\sqrt{3}} \rho \frac{\left(3-\rho^2\right)^2}{2} d\rho.$$

U posljednjem integralu zgodno je uzeti smjenu $3-\rho^2 = t$. Dobija se dalje da je

$\bar{z} = \frac{4}{9}\int_3^0 \frac{t^2}{2} \cdot \left(\frac{-1}{2}\right) dt = \ldots = 1.$ Znači, $T(0,0,1)$.

Napomena: U nekim slučajevima možemo i bez računanja odmah zaključiti da je neka od koordinata težišta jednaka nuli. Radi se o slučajevima kada su jednačine površi koje opisuju oblast Ω simetrične u odnosu na neku od promjenljivih x, y ili z. Tako npr. u posljednjem zadatku, ako

označimo $f(x,y,z) = z - \left(3 - x^2 - y^2\right) = x^2 + y^2 - z - 3$, imamo da je

$f(x,y,z) = f(-x,y,z)$ i $f(x,y,z) = f(x,-y,z)$, što znači da je funkcija

$f(x,y,z)$ simetrična u odnosu na x i u odnosu na y. Zato smo dobili da je

$\bar{x} = \bar{y} = 0.$

Zadaci za samostalan rad:

3. $\Omega : z = \frac{y^2}{2},\ x = 0,\ y = 0,\ z = 0,\ 2x + 3y - 12 = 0.$
4. $x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 = ax.$

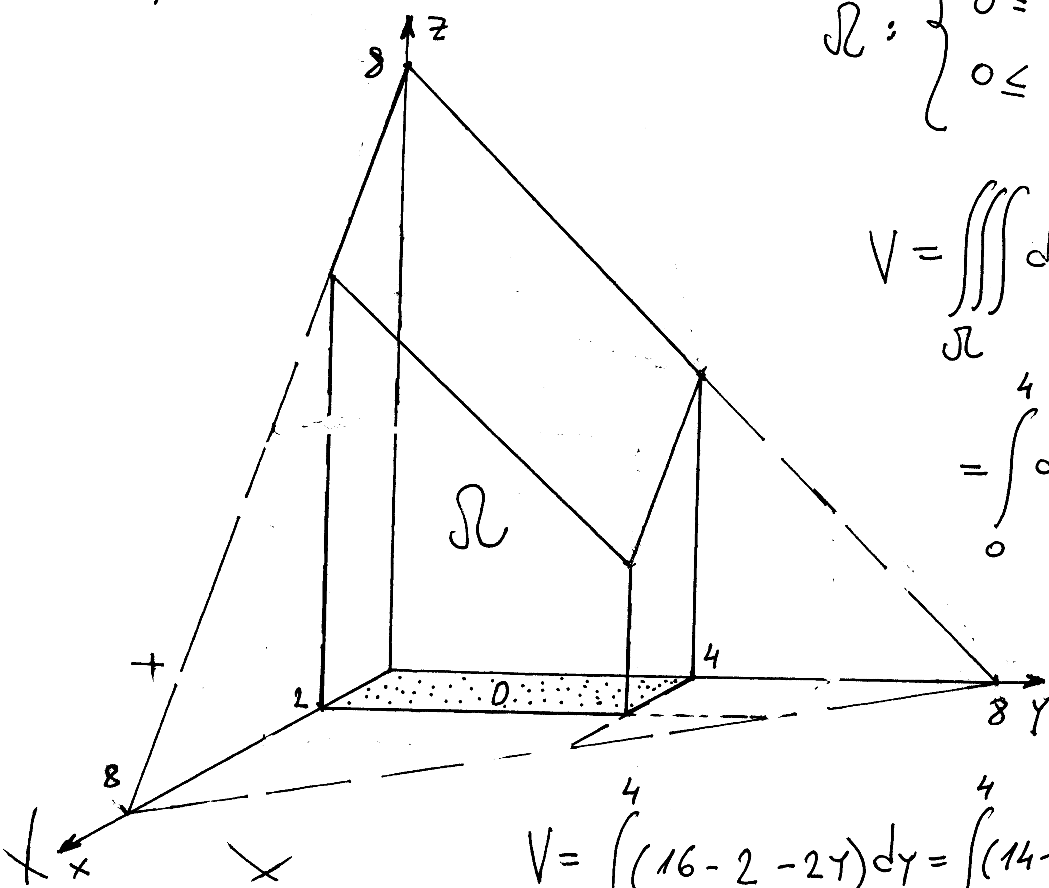
(#) Naći težište homogenog tijela ograničenog sa ravnima $x=0,\ y=0,\ z=0,\ x=2,\ y=4$ i $x+y+z=8$ (koso zasječen paralelopiped).

Rj. Težište $T(x_T, y_T, z_T)$ homogenog tijela ograničenog sa oblašću Ω tražimo po formulama

$$x_T = \frac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz, \quad y_T = \frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz, \quad z_T = \frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz$$

gdje je V zapremina tijela Ω.

Skicirajmo dato tijelo



$$\Omega: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 4 \\ 0 \le z \le 8-x-y \end{cases}$$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{D} (8-x-y)\,dx\,dy$$

$$= \int_0^4 dy \int_0^2 (8-x-y)\,dx =$$

$$= \int_0^4 \left(8x\Big|_0^2 - \frac{1}{2}x^2\Big|_0^2 - yx\Big|_0^2\right)dy$$

$$V = \int_0^4 (16-2-2y)\,dy = \int_0^4 (14-2y)\,dy = 14y\Big|_0^4 - 2\cdot\frac{1}{2}y^2\Big|_0^4$$

$$V = 14\cdot 4 - 16 = 4(14-4) = 40$$

$$V = 40$$

$$\iiint\limits_{\Omega} x\,dx\,dy\,dz = \int_0^4 dy \int_0^2 x\,dx \int_0^{8-x-y} dz = \int_0^4 dy \int_0^2 (8x - x^2 - yx)\,dx = \int_0^4 \left(4x^2\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2 - y\frac{1}{2}x^2\Big|_0^2\right)dy$$

$$= \int_0^4 \left(16 - \frac{8}{3} - 2y\right)dy = \int_0^4 \left(\frac{40}{3} - 2y\right)dy = \frac{40}{3}y\Big|_0^4 - 2\cdot\frac{1}{2}y^2\Big|_0^4 = \frac{160}{3} - 16 = \frac{112}{3}$$

$$\iiint_{\Omega} y\,dx\,dy\,dz = \int_0^2 dx \int_0^4 y\,dy \int_0^{8-x-y} dz = \int_0^2 dx \int_0^4 y(8-x-y)\,dy = \int_0^2 dx \int_0^4 (8y - xy - y^2)\,dy =$$

$$= \int_0^2 \left(8\frac{1}{2}y^2\Big|_0^4 - x\frac{1}{2}y^2\Big|_0^4 - \frac{1}{3}y^3\Big|_0^4 \right) dx = \int_0^2 \left(64 - 8x - \frac{64}{3}\right) dx = \int_0^2 \left(\frac{128}{3} - 8x\right) dx =$$

$$= \frac{128}{3} x\Big|_0^2 - 8\cdot\frac{1}{2}x^2\Big|_0^2 = \frac{256}{3} - 16 = \frac{208}{3}$$

$$\iiint_{\Omega} z\,dx\,dy\,dz = \underset{\ldots}{\text{završiti za vježbu}} = \frac{320}{3}$$

Prema tome, $x_T = \frac{1}{V}\iiint_{\Omega} x\,dx\,dy\,dz = \frac{1}{\cancel{40}} \cdot \frac{\cancel{112}}{3} = \frac{14}{15}$

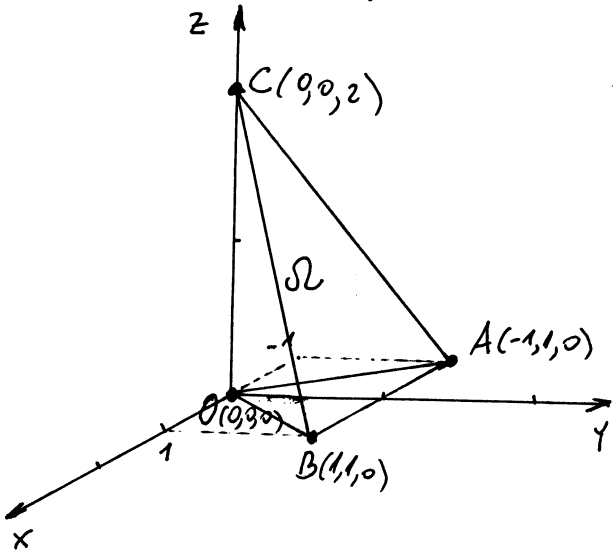
$$y_T = \frac{1}{V}\iiint_{\Omega} y\,dx\,dy\,dz = \frac{1}{\cancel{40}_{5}} \cdot \frac{\cancel{208}^{25}}{3} = \frac{25}{15}$$

$$z_T = \frac{1}{V}\iiint_{\Omega} z\,dx\,dy\,dz = \frac{1}{\cancel{40}_{1}} \cdot \frac{\cancel{320}^{8}}{3} = \frac{8}{3}$$

Težište homogenog tijela je $T\left(\frac{14}{15}, \frac{25}{15}, \frac{8}{3}\right)$.

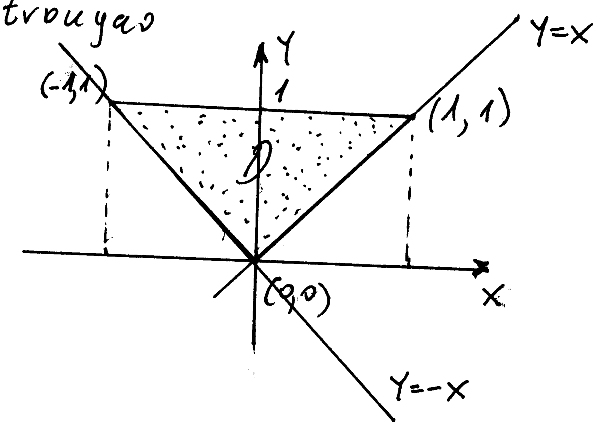
Izračunati pomoću trostrukog integrala zapreminu i težište tetraedra OABC, ako je O(0,0,0), A(-1,1,0), B(1,1,0), C(0,0,2).

Rj: Skicirajmo dato tijelo



$$V=\iiint\limits_{\Omega} dx\,dy\,dz$$

Primjetimo da je projekcija tetraedra na xOy ravan trougao



Odredimo jednačinu ravni kroz tačke A, B i C

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$ jednačina ravni kroz tri tačke

$$\begin{vmatrix} x-(-1) & y-1 & z-0 \\ 1-(-1) & 1-1 & 0-0 \\ 0-(-1) & 0-1 & 2-0 \end{vmatrix} = \begin{vmatrix} x+2 & y-1 & z \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix} = (x+2)\cdot 0-(y-1)\cdot(4-0)+z\cdot(-2-0)$$

$$= -4y+4-2z$$

$$-4y+4-2z=0 \quad /:2$$

$$-2y-z+2=0$$ jednačina ravni koja prolazi kroz tačke A, B i C

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\int_0^1 dy\int_{-y}^{y} dx\int_0^{-2y+2} dz=\int_0^1 dy\int_{-y}^{y} z\Big|_0^{-2y+2} dx=$$

$$=\int_0^1 dy\int_{-y}^{y}(-2y+2)\,dx=\int_0^1\Big(-2yx\Big|_{-y}^{y}+2x\Big|_{-y}^{y}\Big)dy=\int_0^1(-4y^2+4y)\,dy=-4\cdot\frac{1}{3}y^3\Big|_0^1+4\cdot\frac{1}{2}y^2\Big|_0^1=$$

$y-(-y)$

$$=-\frac{4}{3}+2=\frac{2}{3}$$ traženo rješenje

Zadaci za vježbu

Zapremina tela. II

U zadacima 3609 — 3625 pomoću trojnih integrala izračunati zapremine tela ograničenih datim površinama (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3609. Cilindrima $z=4-y^2$ i $z=y^2+2$ i ravnima $x=-1$ i $x=2$.

3610. Paraboloidima $z=x^2+y^2$ i $z=x^2+2y^2$ i ravnima $y=x$, $y=2x$ i $x=1$.

3611. Paraboloidima $z=x^2+y^2$ i $z=2x^2+2y^2$, cilindrom $y=x^2$ i ravni $y=x$.

3612. Cilindrima $z=\ln(x+2)$ i $z=\ln(6-x)$ i ravnima $x=0$, $x+y=2$ i $x-y=2$.

3613*. Paraboloidom $(x-1)^2+y^2=z$ i ravni $2x+z=2$.

3614*. Paraboloidom $z=x^2+y^2$ i ravni $z=x+y$.

3615*. Sferom $x^2+y^2+z^2=4$ i paraboloidom $x^2+y^2=3z$.

3616. Sferom $x^2+y^2+z^2=R^2$ i paraboloidom $x^2+y^2=R(R-2z)$ $(z\geqslant 0)$.

3617. Paraboloidom $z=x^2+y^2$ i konusom $z^2=xy$.

3618. Sferom $x^2+y^2+z^2=4Rz-3R^2$ i konusom $z^2=4(x^2+y^2)$ (misli se na deo loptine zapremine koji leži unutar konusa).

3619*. $(x^2+y^2+z^2)^2=a^3x$.

3620. $(x^2+y^2+z^2)^2=axyz$.

3621. $(x^2+y^2+z^2)^3=a^2z^4$. **3622.** $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$,

3623. $(x^2+y^2+z^2)^3=a^2(x^2+y^2)^2$.

3624. $(x^2+y^2)^2+z^4=a^3z$.

3625. $x^2+y^2+z^2=1$, $x^2+y^2+z^2=16$, $z^2=x^2+y^2$, $x=0$, $y=0$, $z=0$ $(x>0,\ y>0,\ z\geqslant 0)$.

Rješenja

3609. 8.

3610. $\dfrac{7}{12}$. **3611.** $\dfrac{3}{35}$.

3612. $4(4-3\ln 3)$.

3613*. $\dfrac{\pi}{2}$. Projekcija tela na ravan xOy je krug.

3614. $\dfrac{\pi}{8}$. Preneti koordinatni početak u tačku $\left(\dfrac{1}{2}, \dfrac{1}{2}, 0\right)$.

3615*. $\dfrac{19}{6}\pi$ i $\dfrac{15}{2}\pi$. Preći na cilindrične koordinate.

3616. $\dfrac{5}{12}\pi R^3$. **3617.** $\dfrac{\pi}{96}$.

3618. $\dfrac{92}{75}\pi R^3$.

3619*. $\dfrac{1}{3}\pi a^3$. Preći na sferne koordinate.

3620. $\dfrac{a^3}{360}$. **3621.** $\dfrac{4}{21}\pi a^3$.

3622. $\dfrac{4}{3}\pi a^3$. **3623.** $\dfrac{64}{105}\pi a^3$.

3624. $\dfrac{\pi^2a^3}{6}$. **3625.** $\dfrac{21(2-\sqrt{2})}{4}\pi$.

Težišta homogenih tela

U zadacima 3666 — 3672 naći težišta homogenih tela ograničenih datim površinama.

3666. Ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasečeni paralelepiped).

3667. Elipsoidom $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ i koordinatnim ravnima (misli se na deo elipsoida koji leži u prvom oktantu).

3668. Cilindrom $z=\dfrac{y^2}{2}$ i ravnima $x=0$, $y=0$, $z=0$ i $2x+3y-12=0$.

3669. Cilindrima $y=\sqrt{x}$, $y=2\sqrt{x}$ i ravnima $z=0$ i $x+z=0$.

3670. Paraboloidom $z=\dfrac{x^2+y^2}{2a}$ i sferom $x^2+y^2+z^2=3a^2$ $(z>0)$.

3671. Sferom $x^2+y^2+z^2=R^2$ i konusom $z\,\mathrm{tg}\,\alpha=\sqrt{x^2+y^2}$ (loptin isečak).

3672. $(x^2+y^2+z^2)^2=a^3z$.

Rješenja

3666. $\xi=\dfrac{14}{15}$, $\eta=\dfrac{26}{15}$, $\zeta=\dfrac{8}{3}$. **3667.** $\xi=\dfrac{3}{8}a$, $\eta=\dfrac{3}{8}b$, $\zeta=\dfrac{3}{8}c$.

3668. $\xi=\dfrac{6}{5}$, $\eta=\dfrac{12}{5}$, $\zeta=\dfrac{8}{5}$. **3669.** $\xi=\dfrac{18}{7}$, $\eta=\dfrac{15}{16}\sqrt{6}$, $\zeta=\dfrac{12}{7}$.

3670. $\xi=0$, $\eta=0$, $\zeta=\dfrac{5a}{83}(6\sqrt{3}+5)$.

3671. $\xi=0$, $\eta=0$, $\zeta=\dfrac{3R}{8}(1+\cos\alpha)$. **3672.** $\xi=0$, $\eta=0$, $\zeta=\dfrac{9a}{20}$.